Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

On 3d Irreducible and Indecomposable Polyhedra and The Number of Interior Steiner Points

Hang Si and Nadja Goerigk

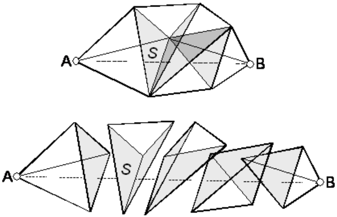
Tetrahedron V - Liége - 2016.07.05

Mohrenstrasse 39 · 10117 Berlin · Germany · Tel. +49 30 20372 0 · www.wias-berlin.de
July 5, 2016

Content

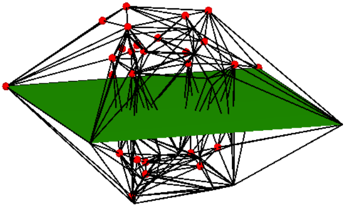
Motivation – Constrained Tetrahedralizations

How to generate a tetrahedralization that contains a set of constraints, i.e., edges and (triangular or polygonal) faces?



A constrained edge AB is missing

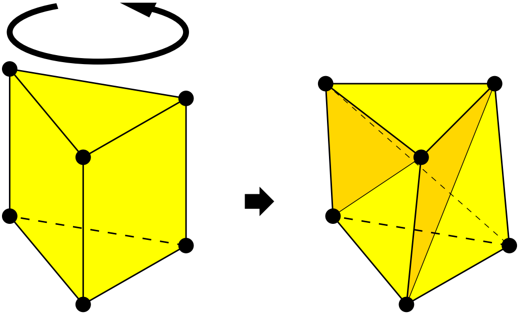
Image from [**Owen 1999**]



A constrained face (in green) is missing

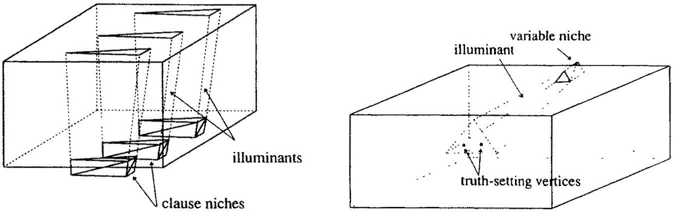
3D Indecomposable Polyhedra

There are 3d simple polyhedra which cannot be tetrahedralized without extra vertices.



The Schönhardt Polyhedron [**1928**]

Ruppert & Seidel [1993]: It is NP-complete to decide whether a given 3D polyhedron can be triangulated without using additional points.



Ruppert & Seidel's Polyhedron [**1993**]

How many Steiner points are necessary?

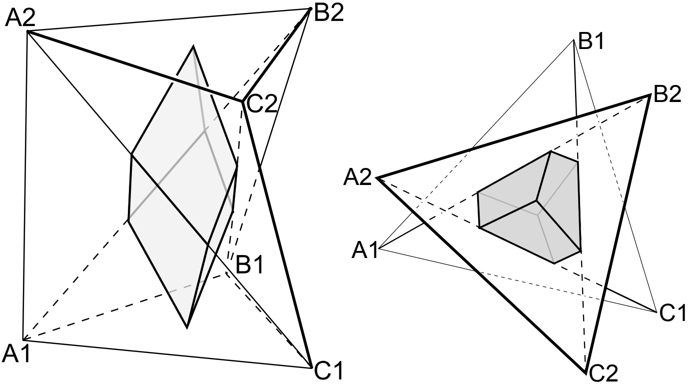
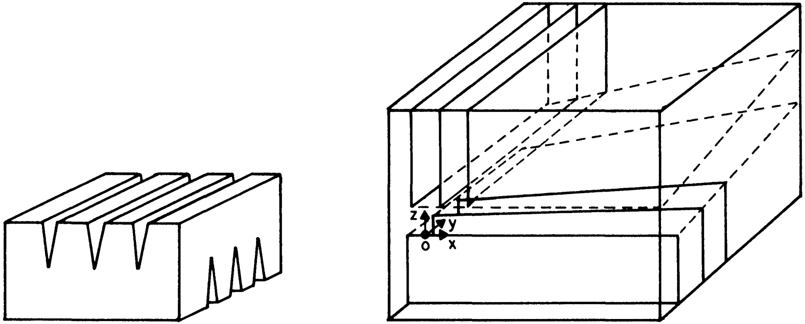


Figure: The (open) valid domain for placing Steiner points inside the Schönhardt polyhedron. A side view (left) and a top view (right) are shown.

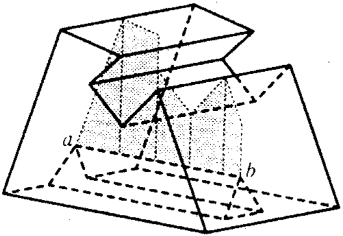
The lower bound is $\Omega(n^2)$



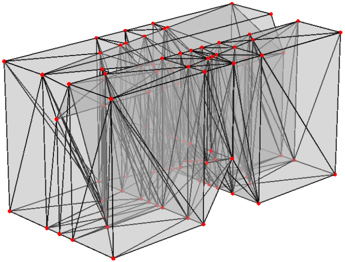
The Chazelle's polyhedron [**1984**]

Convex Decomposition Algorithms

- **Chazelle & Palios** [**1990**]: A non-convex polyhedron of zero genus with n vertices and r reflex edges can be decomposed into $O(n + r^2)$ tetrahedra. .
- **Bajaj and Dey** [**1992**]: A non-convex polyhedron of zero genus with n vertices and r reflex edges can be decomposed into $O(nr^2 + r^{7/2})$ tetrahedra in $O(nr + r^{5/2})$ space.



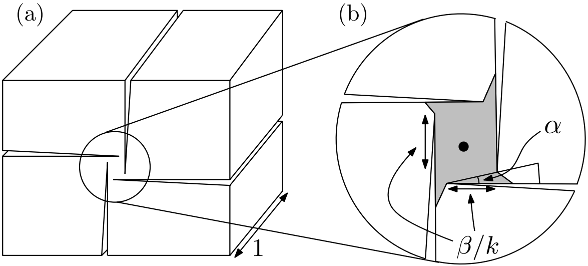
The Fence-Off algorithm
from [**Chazelle & Palios 1990**]



A test result of the algorithm
implemented in [**Palios 1992**]

Convex Decomposition Algorithms

- **Erickson** [**2005**]: Every *local polyhedra* can be decomposed into $O(n \log n)$ tetrahedra.
- **De Berg & Gray** [**2010**]: Every locally-fat polyhedron with convex fat faces can be decomposed into $O(n)$ tetrahedra.
- Both local polyhedra and locally-fat polyhedra also include polyhedra which are tetrahedralizable without Steiner points.



A locally-fat polyhedron with fat faces whose interior cannot be covered by a bounded number of fat tetrahedra. [**De Berg & Gray 2010**]

Conforming Delaunay Tetrahedralizations

- Algorithms: [**Murphy, Mount, & Gable 2000**], [**Cohen-Steiner, Colin de Verdière, & Yvinec 2002**]
- A conforming Delaunay tetrahedralization may include a lot of Steiner points. An $O(n^3)$ upper bound of Steiner points for 2d conforming Delaunay triangulation is proven [**Edelsbrunner & Tan 1993**]. (A recent improvement of this result by Bishop is $O(n^{2.5})$). The 3d case is still open.

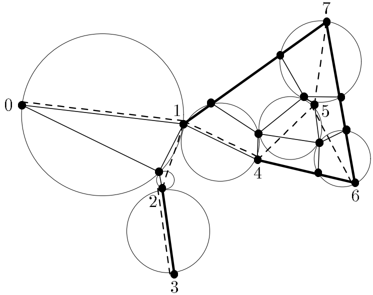
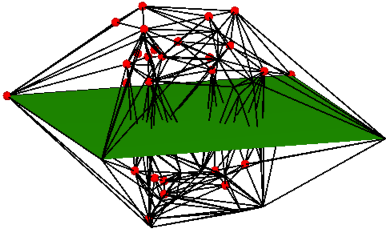


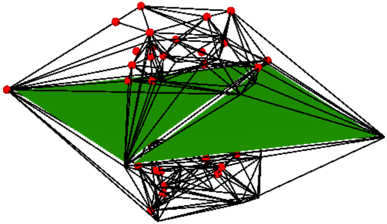
Figure from [**Edelsbrunner & Tan 1993**]

Constrained Delaunay Tetrahedralization

- Algorithms: [**Shewchuk 2002, 2003**], [**Si & Gärtner 2005**], [**Si & Shewchuk 2012**].
- The number of Steiner points is (significantly) reduced. However, an upper bound is still unknown.



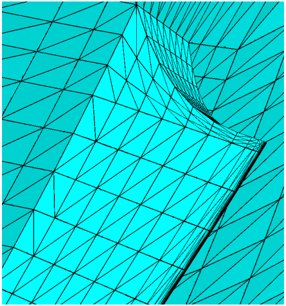
A Delaunay tetrahedralization

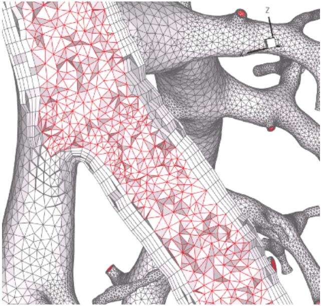


A Constrained DT.

Interior Steiner Points

- In many applications, the input boundary are required to be preserved.
- Steiner points (if they are necessary) can only be placed in the interior of the domain.
- Neither conforming nor constrained Delaunay terahedralization can satisfy this requirement.





Boundary Recovery Methods

- **George, Hecht, & Saltel** [**1991**]: Use edge/face swaps together with interior Steiner points insertion.
- **Weatherill & Hassan** [**1994**] Insert Steiner points at where the boundaries and $\mathcal{T}$ intersect, delete vertices or relocate them from the boundaries afterwards.
- **George, Borouchaki, & Saltel** [**2003**]: Combine the above two methods.

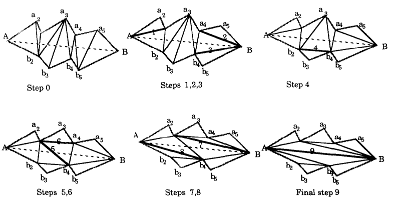


Fig. 11. Step 0, steps 1, 2, 3, step 4, steps 5, 6, steps 7, 8, and step 9.

George, Hecht, and Saltel 1991

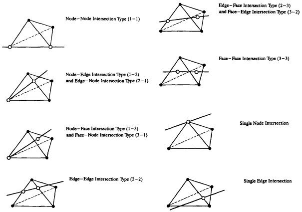
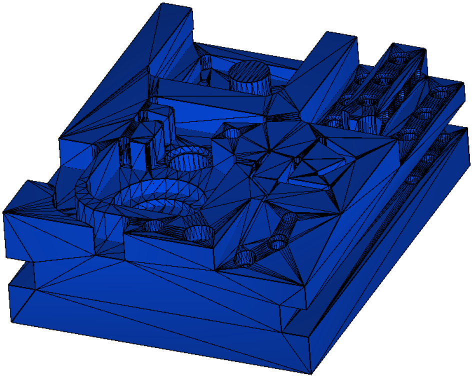


Figure 5. Types of edge-tetrahedron intersections for missing surface edges

Weatherill and Hassan 1994

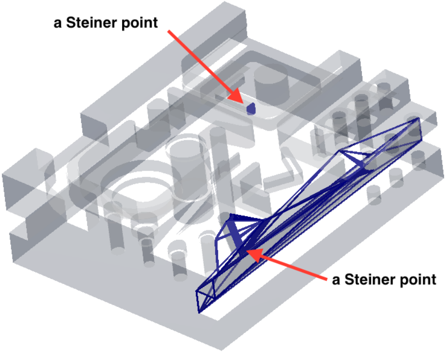
Experiment 1 (`TetGen v1.5`)



Example: `mohne` (from INRIA Mesh Repository)

Input: 2760 points, 5560 triangles
Output: added 2 Steiner points

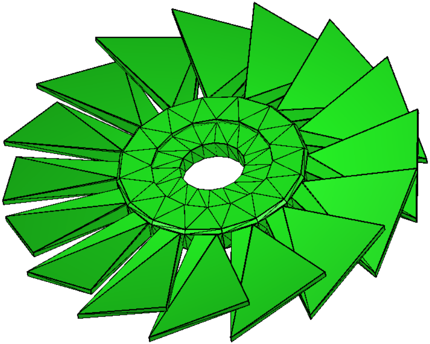
Experiment 1 (`TetGen v1.5`)



Example: `mohne` (from INRIA Mesh Repository)

Input: 2760 points, 5560 triangles
Output: added 2 Steiner points

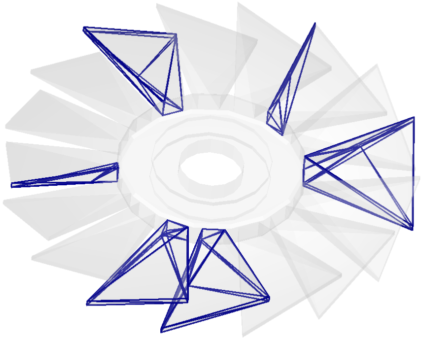
Experiment 2 (`TetGen v1.5`)



Example: `03-machinery-part_cut` (from INRIA Mesh Repository)

Input: 448 points, 1120 triangles
Output: added 8 Steiner points

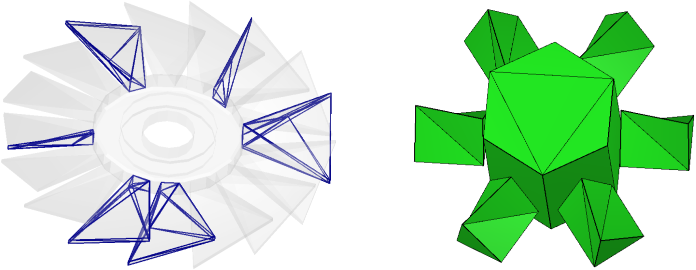
Experiment 2 (`TetGen v1.5`)



Example: `03-machinery-part_cut` (from INRIA Mesh Repository)

Input: 448 points, 1120 triangles
Output: added 8 Steiner points

Experiment 2 (`TetGen v1.5`)



Example: `03-machinery-part_cut` (from INRIA Mesh Repository)

Input: 448 points, 1120 triangles
Output: added 8 Steiner points

Bagemihl's Theorem [1948]

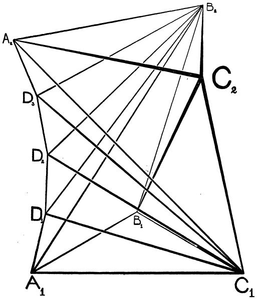
If n is an integer not less than 6, then there exists a polyhedron, π_n, with n vertices and the following properties:

(I) π_n is simple and every one of its faces is a triangle.

(II) If τ is a tetrahedron, each of whose vertices is a vertex of π_n, then not every interior point of τ is an interior point of π_n.

(III) Every open segment whose endpoints are vertices of π_n, but which is not an edge of π_n, lies wholly exterior to π_n.

(IV) Every triangle whose sides are edges of π_n is a face of π_n.



The Bagemihl polyhedron (π_9) with 9 vertices

- The condition (II) follows from (I), (III), and (IV).

Irreducible and Indecomposable Polyhedra

Definition: A 3d (non-convex) polyhedron P is an irreducible and indecomposable polyhedron if for every tetrahedron τ, whose vertices is a vertex of P, not every interior point of τ is an interior point of P.

Irreducible and Indecomposable Polyhedra

Definition: A 3d (non-convex) polyhedron P is an irreducible and indecomposable polyhedron if for every tetrahedron τ, whose vertices is a vertex of P, not every interior point of τ is an interior point of P.

Remarks:

- If a polyhedron is irreducible then it is indecomposable, but the reverse is false.
- Bagemihl's Theorem claims there exist a family of irreducible and indecomposable polyhedra.

Irreducible and Indecomposable Polyhedra

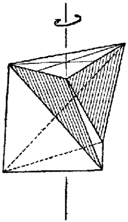
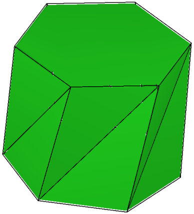
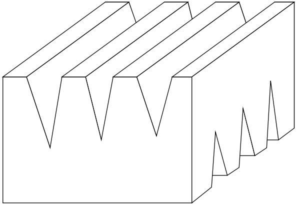
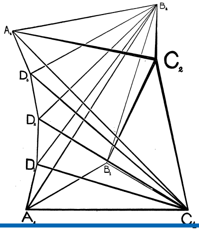
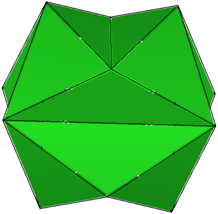
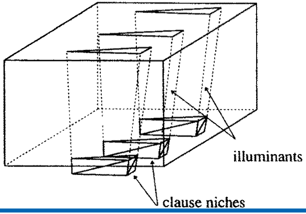
Definition: A 3d (non-convex) polyhedron P is an irreducible and indecomposable polyhedron if for every tetrahedron τ, whose vertices is a vertex of P, not every interior point of τ is an interior point of P.

Remarks:

- If a polyhedron is irreducible then it is indecomposable, but the reverse is false.
- Bagemihl's Theorem claims there exist a family of irreducible and indecomposable polyhedra.

Proposition: If a 3d simple and simplical polyhedron contains no open segments, then it is reducible. (to be proven)

Some Well-Known Polyhedra

Schönhardt polyhedron **1928**	Rambau's generalisation **2005**	Chazelle's polyhedron **1984**
		
Bagemihl's generalisation **1948**	Jesson's polyhedron **1967**	Rupper & Seidel **1993**
		

Extended Bagemihl's Theorem

Theorem [Goerigk & Si 2015] If n is an integer not less than 6, then there exists an irreducible and indecomposable polyhedron, σ_n, with n vertices and the following properties:

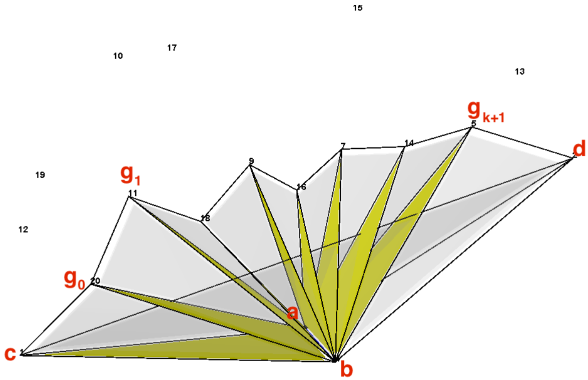
(I) σ_n is simple and every one of its faces is a triangle.

(II) **Every open segment e, whose endpoints are vertices of σ_n, but which is not an edge of σ_n, does not lie in the interior of σ_n.**

(III) Every triangle whose sides are edges of σ_n is a face of σ_n.

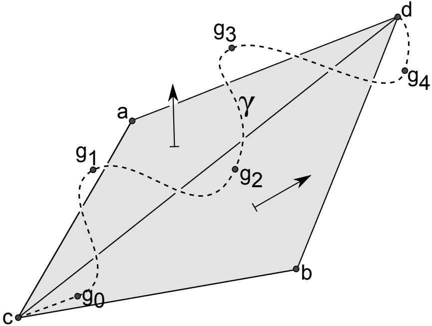
A Motivation Example

A missing edge $[c, d]$ is crossing a number of triangles that all share a common line segment $[a, b]$.



Construction of σ_n

Choose four non-coplanar points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$, and a (simple) curve γ starting at $\mathbf{c}$ and ending at $\mathbf{d}$, and γ lies in the intersection of the two open halfspaces bounded by the triangles $\mathbf{cda}$ and $\mathbf{dcb}$ (using the right-hand rule to orient the vertices of the triangles).
Now we will choose $k+2$ ($k \geq 0$) distinct points, denoted as $\mathbf{g_0}, \ldots, \mathbf{g_{k+1}}$, on the curve γ from $\mathbf{c}$ to $\mathbf{d}$.



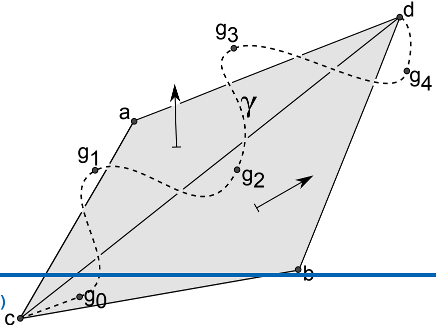
Construction of σ_n

(c1) The line segment $\mathbf{cd}$ intersects all the triangles $\mathbf{abg}_i,\ i = 0, \ldots, k+1$.

(c2) Given two adjacent points $\mathbf{g}_i$ and $\mathbf{g}_{i+1}$, for $i = 0, \ldots, k$, on the curve γ, the point $\mathbf{g}_{i+1}$ and $\mathbf{d}$ must lie in the same halfspace bounded by the plane containing $\mathbf{abg}_i$.

(c3) Let $\mathbf{g}_i$ and $\mathbf{g}_j$, for $i, j = -1, \ldots, k+2$ and $i \neq j$, be two non-adjacent points on the curve γ where $\mathbf{g}_{-1} := \mathbf{c}$ and $\mathbf{g}_{k+2} := \mathbf{d}$. Without loss of generality, assume $i < j$. Then the line segment $\mathbf{g}_i\mathbf{g}_j$ (except $\mathbf{g}_{-1}\mathbf{g}_{k+2} = \mathbf{cd}$) does not intersect all triangles $\mathbf{abg}_l$, where $i < l < j$.

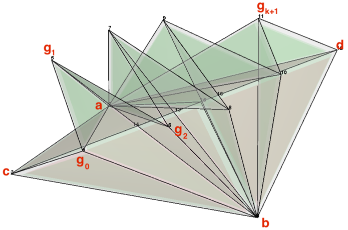
(c4) Let $\mathbf{g}_i$, $\mathbf{g}_{i+1}$ and $\mathbf{g}_{i+2}$, for $i = -1, \ldots, k$, be three consecutive points on the curve γ. Then the three points are neither coplanar with $\mathbf{a}$ nor $\mathbf{b}$.

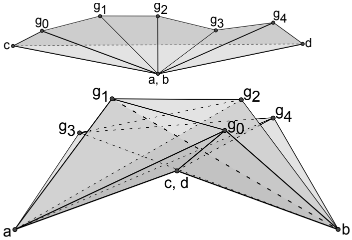


Construction of σ_n

Now the polyhedron σ_n, $n = 6 + k$, where $k \geq 0$, is constructed by choosing the boundary faces listed in Table.

(1)	$(\mathbf{a}, \mathbf{c}, \mathbf{d}), (\mathbf{b}, \mathbf{c}, \mathbf{d})$
(2)	$(\mathbf{a}, \mathbf{c}, \mathbf{g}_0), (\mathbf{b}, \mathbf{c}, \mathbf{g_0}), (\mathbf{a}, \mathbf{d}, \mathbf{g}_{k+1}), (\mathbf{b}, \mathbf{d}, \mathbf{g}_{k+1})$
(3)	$(\mathbf{a}, \mathbf{g}_i, \mathbf{g}_{i+1}), (\mathbf{b}, \mathbf{g}_i, \mathbf{g}_{i+1})$, where $i = 0, \ldots, k$





Properties of σ_n

- σ_n satisfies the extended Bagemihl's Theorem, hence it is an irreducible indecomposable polyhedron.
- σ_n is combinatorially the same as π_n, in particular, $\sigma_6 = \pi_6$ and is the Schönhardt polyhedron.

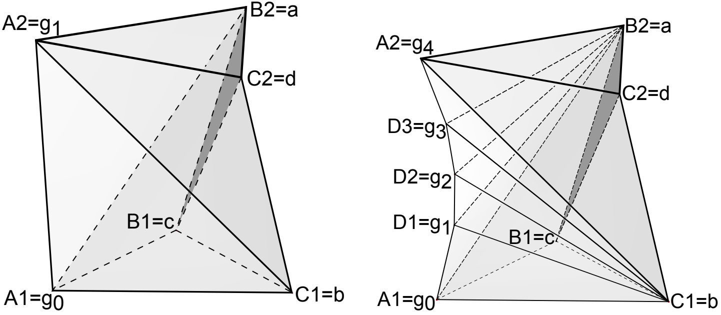
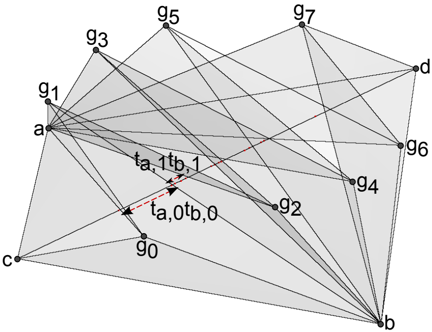


Figure: The mapping between the vertices of σ_n and the vertices of the Bagemihl polyhedron π_n.

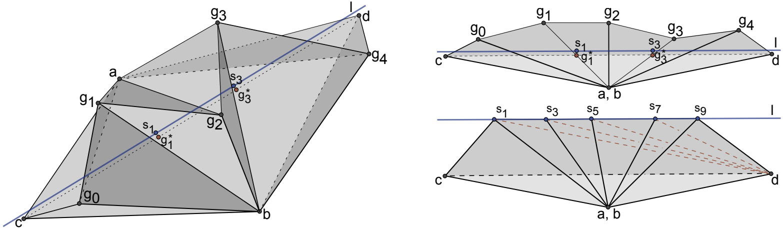
The Number of Interior Steiner Points

A σ_n may need more than one interior Steiner point to be decomposed.



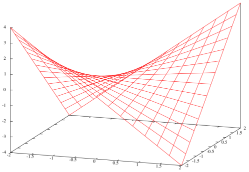
The Number of Interior Steiner Points

Theorem [**Goerigk & Si**] σ_n can be tetrahedralized by adding $\left\lceil \frac{n-5}{2} \right\rceil$ interior Steiner points.



The Chazelle Polyhedron

The non-convex polyhedron constructed by Chazelle, known as the Chazelle polyhedron, establishes a quadratic lower bound on the minimum number of convex pieces for the 3d polyhedron partitioning problem.



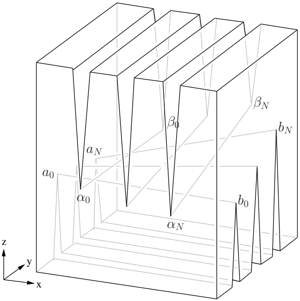
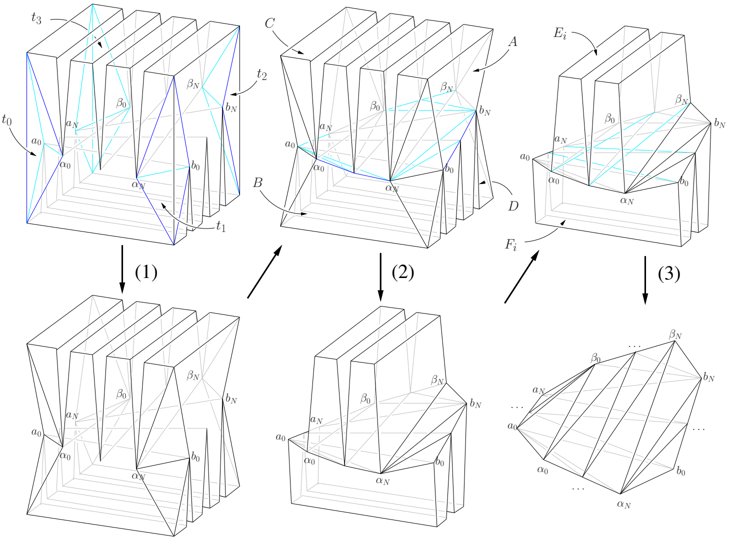


Figure: Left: A saddle surface (a hyperbolic paraboloid). Right: The Chazelle polyhedron with three notches, i.e., $N = 2$, on the top and the bottom faces, respectively.

The Reduced Chazelle Polyhedron



The Reduced Chazelle Polyhedron

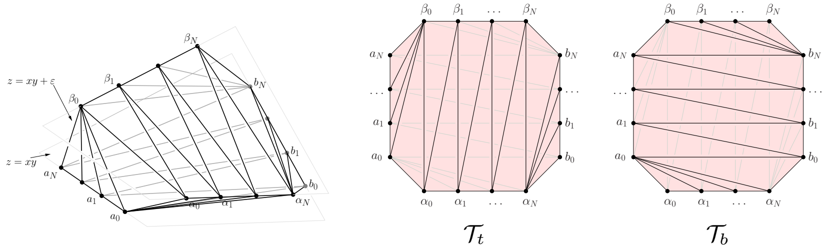


Figure: Left: A reduced Chazelle polyhedron $\Phi_{3,\varepsilon}$. Right: The top triangulation $\mathcal{T}_t$ includes the set of top faces as viewed from the point $(0, 0, +\infty)$ toward the $-z$ direction. The bottom triangulation $\mathcal{T}_b$ includes the set of bottom faces viewed from the point $(0, 0, -\infty)$ toward the $+z$ direction.

Edges Flips and Tetrahedralizations

Sleator et al [**1988**] showed the correspondence between a sequence of edge flips and a tetrahedralisation of a 3d convex polyhedron.

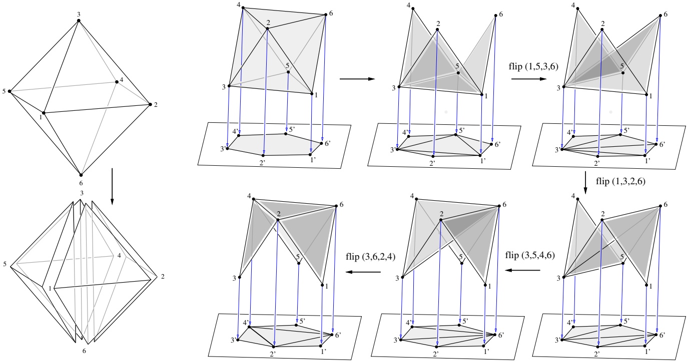
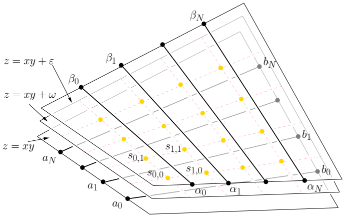


Figure: Left: A tetrahedralisation of an octahedron with four tetrahedra. Right: A sequence of edge flips which corresponds to the tetrahedralisation on the left.

A Placement of Interior Steiner Points



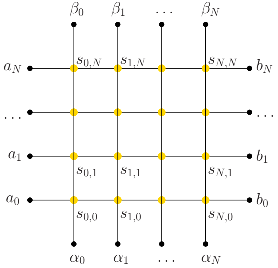


Figure: The interior Steiner points, $\{s_{i,j} \mid i, j = 0, \dots, N\}$, are placed directly at the intersections of the two set of lines in the xy-plane and all lie on the saddle surface $z = xy + \omega$, where $0 < \omega < \varepsilon$.

A Modified Polyhedron

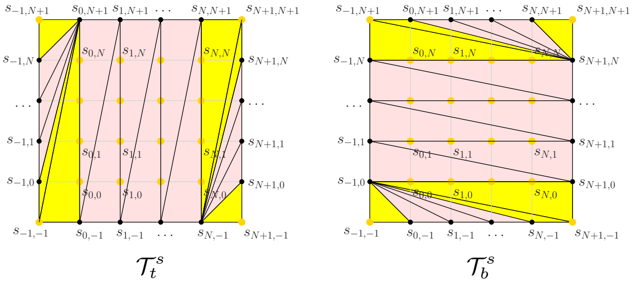


Figure: The modified top and bottom triangulations of $\Phi^s_{N,\varepsilon}$. There are four new vertices: $s_{-1,-1}$, $s_{N+1,-1}$, $s_{-1,N+1}$, and $s_{N+1,N+1}$. The newly added triangles are shown in yellow.

Tetrahedrqlization Step 1: Edge Splitting

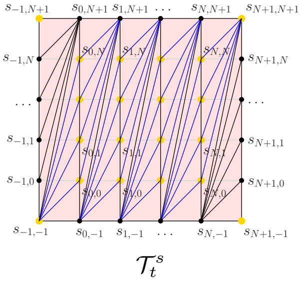


Figure: An example result of the first step of the transformation algorithm. All Steiner points are inserted by splitting the edges.

Tetrahedrqlization Step 2: Edge Flips

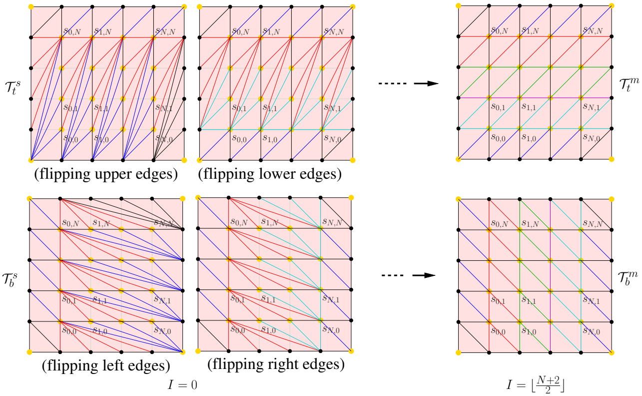


Figure: An example result of the second step of the transformation algorithm. Two sequences of edge flips are applied on top and bottom triangulations, respectively. The resulting two triangulations $\mathcal{T}_t^m$ and $\mathcal{T}_b^m$ are shown on the right.

The Sequence of Edge Flips

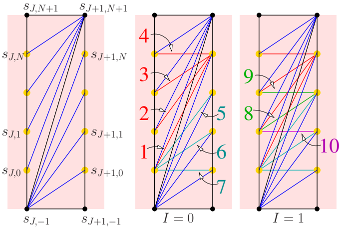
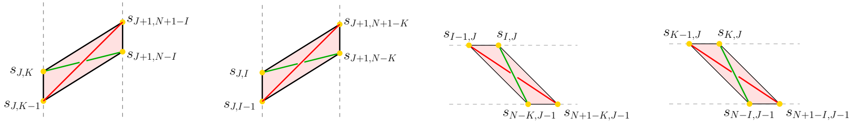


Figure: An example of the sequence of edge flips applied on one section of the top triangulations $\mathcal{T}_t^s$. Left is the initial triangulation before the edge flips. Right shows the sequence is the newly created edges with their indices by the flip sequence.

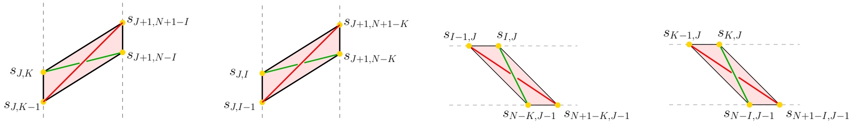
The Four Types of Edge Flips



(1) in $\mathcal{T}_t^s$ upper (2) in $\mathcal{T}_t^s$ lower (3) in $\mathcal{T}_b^s$ left (4) in $\mathcal{T}_b^s$ right

Figure: The four types of edge flips in the algorithm. In these figures, red edges are the input edges, green edges are the resulting edges. Each pair of red and green edges forms a tetrahedron in the interior.

The Four Types of Edge Flips



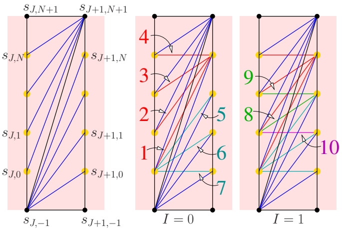
(1) in $\mathcal{T}_t^s$ upper (2) in $\mathcal{T}_t^s$ lower (3) in $\mathcal{T}_b^s$ left (4) in $\mathcal{T}_b^s$ right

Figure: The four types of edge flips in the algorithm. In these figures, red edges are the input edges, green edges are the resulting edges. Each pair of red and green edges forms a tetrahedron in the interior.

Lemma: Let $\det(s_1, s_2, s_3, s_4)$ denote the determinant of the four points $s_1, \ldots, s_4 \in \mathbb{R}^3$. The following determinants on the set of Steiner points are all constant.

$$\begin{aligned}
\det(s_{J+1,N+1-I}, s_{J,K-1}, s_{J+1,N-I}, s_{J,K}) &\equiv 1 \\
\det(s_{J,I-1}, s_{J+1,N+1-K}, s_{J,I}, s_{J+1,N-K}) &\equiv 1 \\
\det(s_{I-1,J}, s_{N+1-K,J-1}, s_{I,J}, s_{N-K,J-1}) &\equiv -1 \\
\det(s_{N+1-I,J-1}, s_{K-1,J}, s_{N-I,J-1}, s_{K,J}) &\equiv -1
\end{aligned} \tag{1}$$

The Four Types of Edge Flips

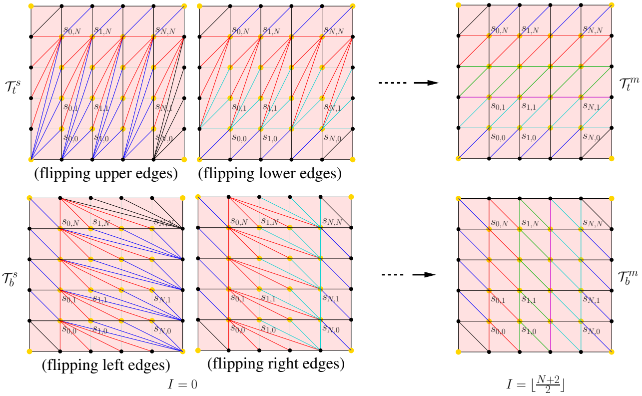


Lemma: Let $\det(s_1, s_2, s_3, s_4)$ denote the determinant of the four points $s_1, \ldots, s_4 \in \mathbb{R}^3$. The following determinants on the set of Steiner points are all constant.

$$\begin{aligned}
\det(s_{J+1,N+1-I}, s_{J,K-1}, s_{J+1,N-I}, s_{J,K}) &\equiv 1 \\
\det(s_{J,I-1}, s_{J+1,N+1-K}, s_{J,I}, s_{J+1,N-K}) &\equiv 1 \\
\det(s_{I-1,J}, s_{N+1-K,J-1}, s_{I,J}, s_{N-K,J-1}) &\equiv -1 \\
\det(s_{N+1-I,J-1}, s_{K-1,J}, s_{N-I,J-1}, s_{K,J}) &\equiv -1
\end{aligned} \tag{1}$$

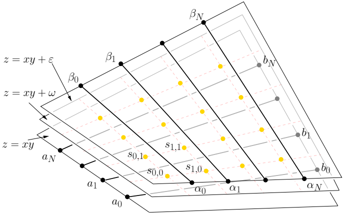
Proof of Correctness

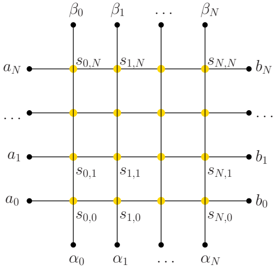
Theorem There exists a tetrahedralisation of $\Phi^s_{N,\varepsilon}$ with the set $\mathcal{S}$ of interior Steiner points.



The Number of Interior Steiner Points

Theorem The reduced Chazelle polyhedron $\Phi^s_{N,\varepsilon}$ needs $(N+1)^2$ interior Steiner points as $\varepsilon \to 0$.





Summary

- We introduced a class of 3d non-convex polyhedra, so-called *irreducible and indecomposable* polyhedra. They are the core object of any 3d indecomposable polyhedron.
- We showed two classes of such polyhedra by generalising the known examples of Bagemihl polyhedra and Chazelle polyhedra.
- The optimal number of interior Steiner points for these polyhedra are proven.

- There exists a family of irreducible and indecomposable polyhedra. We only know few of them. It is interested to further study and construct them.
- How to apply our results to mesh generation, in particular, the 3d boundary recovery problem? One hint of our result is that a good choice of the locations of Steiner points is at the spatial cross of two constraining line segments.
- What are the relations between 3d irreducible and indecomposable polyhedra and 3d non-regular triangulations?