Shape, Homology, Persistence, and Stability

Tetrahedron V 2016

Herbert Edelsbrunner

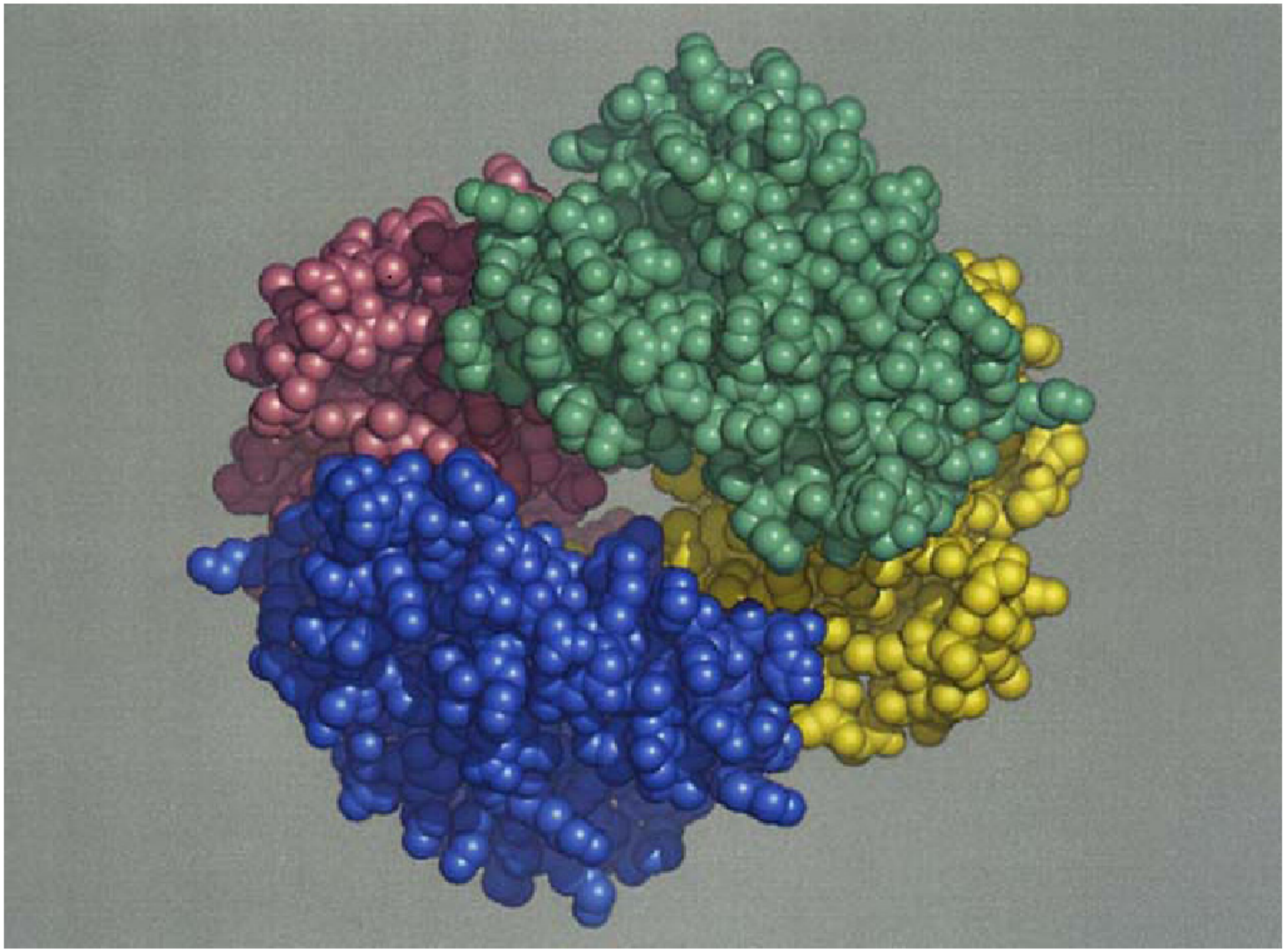
IST Austria

Shape, Homology, Persistence, and Stability

From Proteins to Simplicial Complexes

Hemoglobin

Oxygen transport



FROM PROTEINS TO SIMPLICIAL COMPLEXES

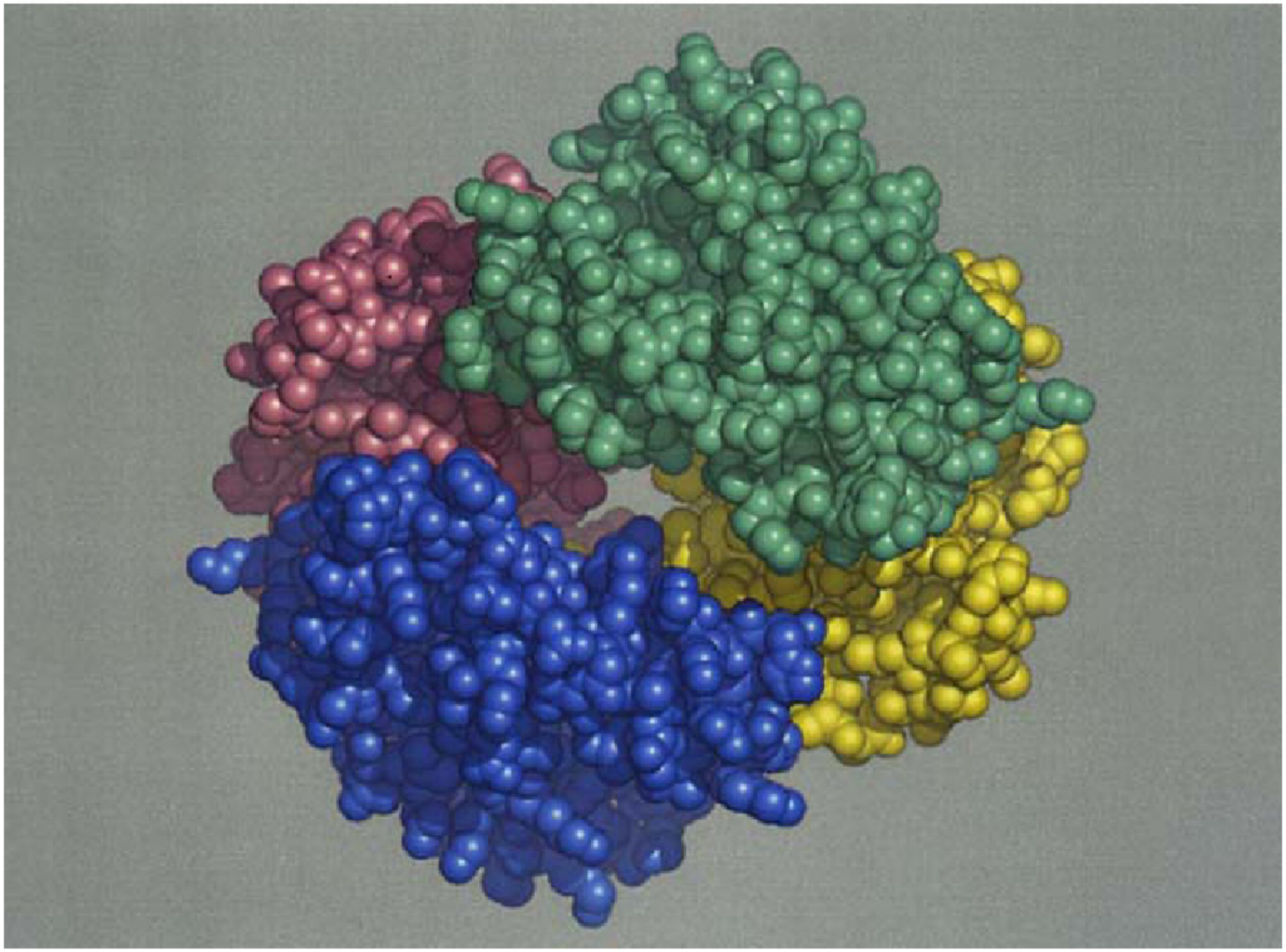
HEMOGLOBIN

OXYGEN TRANSPORT

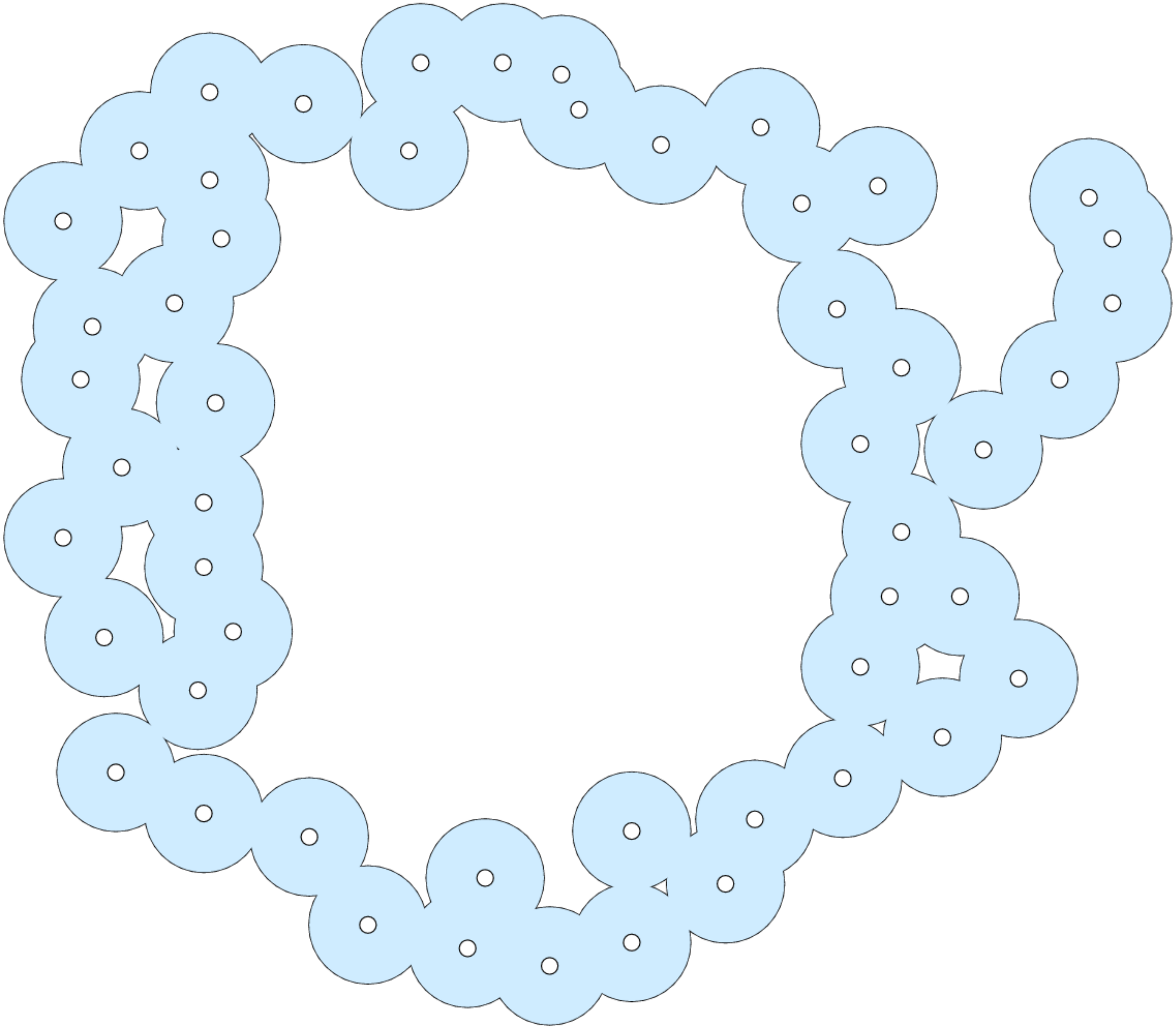
protein $= \bigcup$ balls in $\mathbb{R}^3$

$\Big\downarrow$ Voronoi + nerve

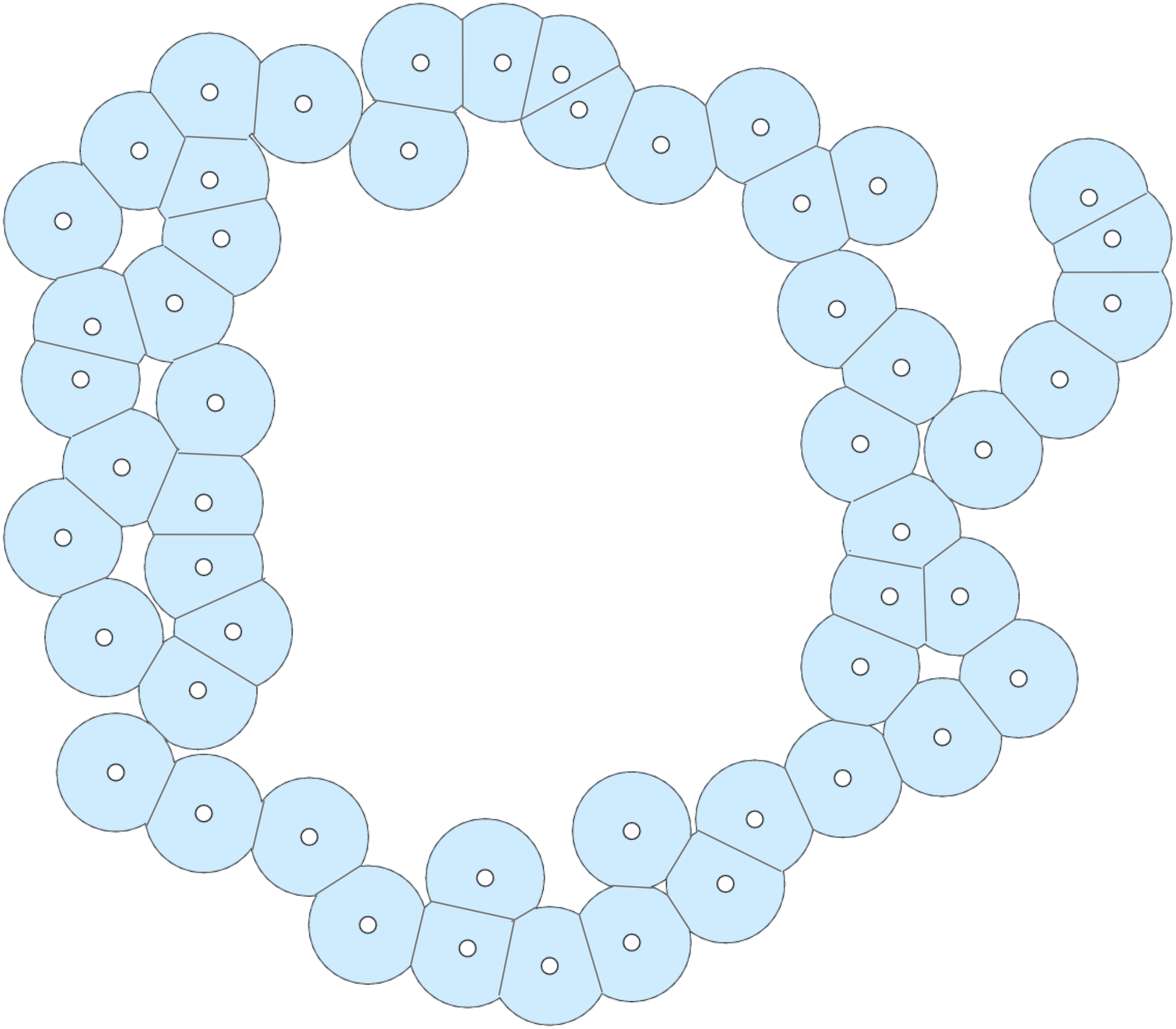
α-complex



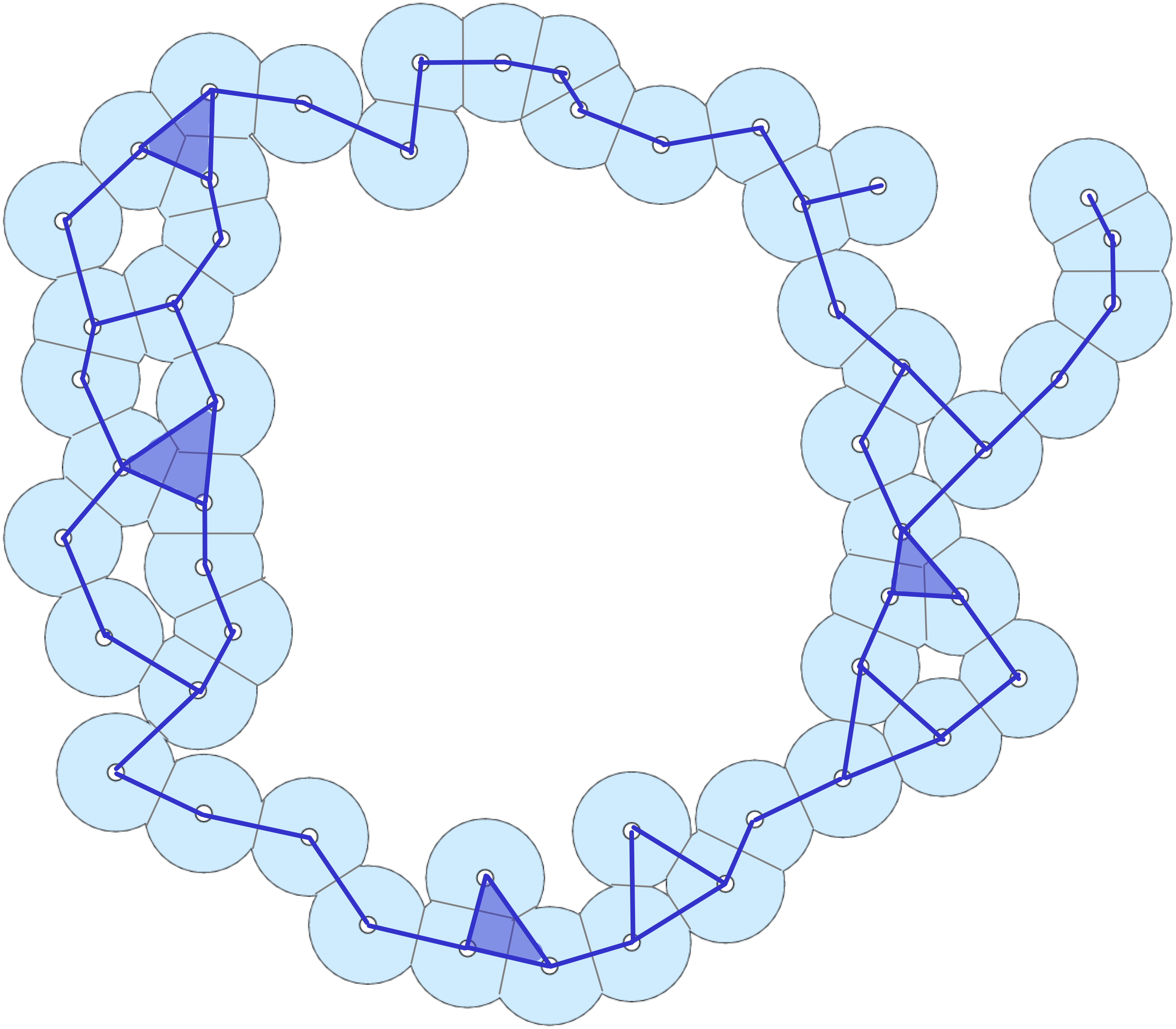
From Proteins to Simplicial Complexes



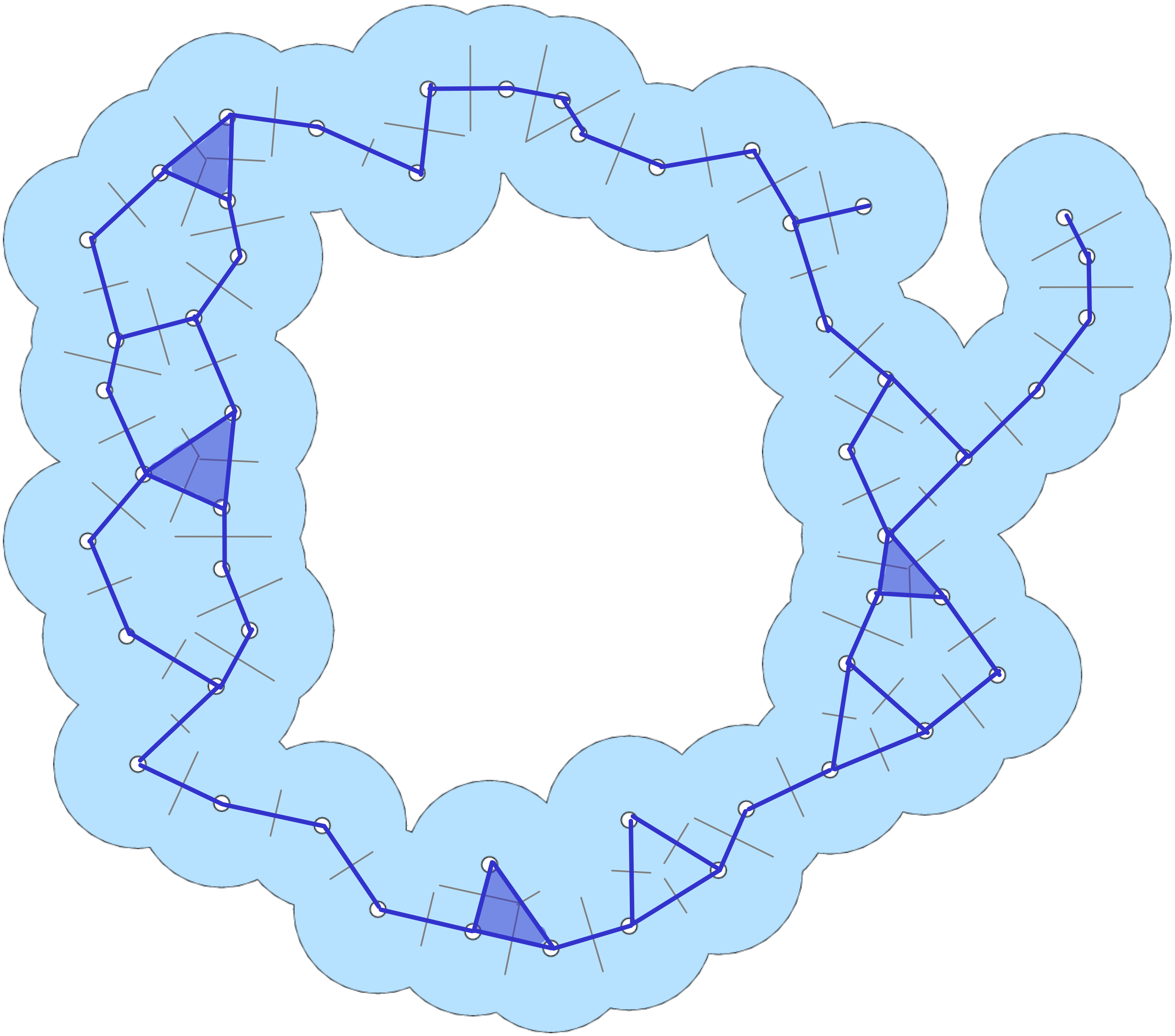
From Proteins to Simplicial Complexes



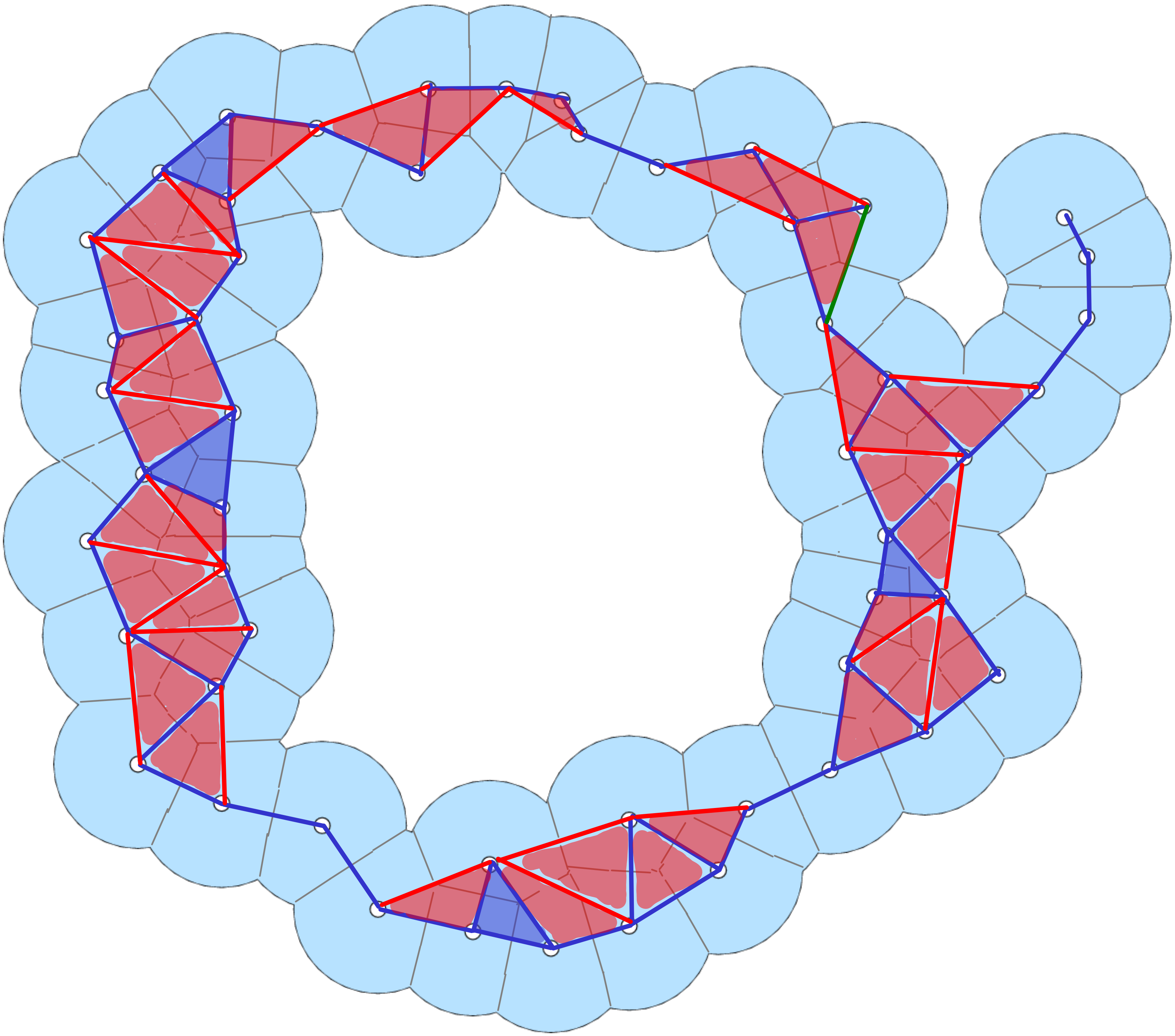
From Proteins to Simplicial Complexes



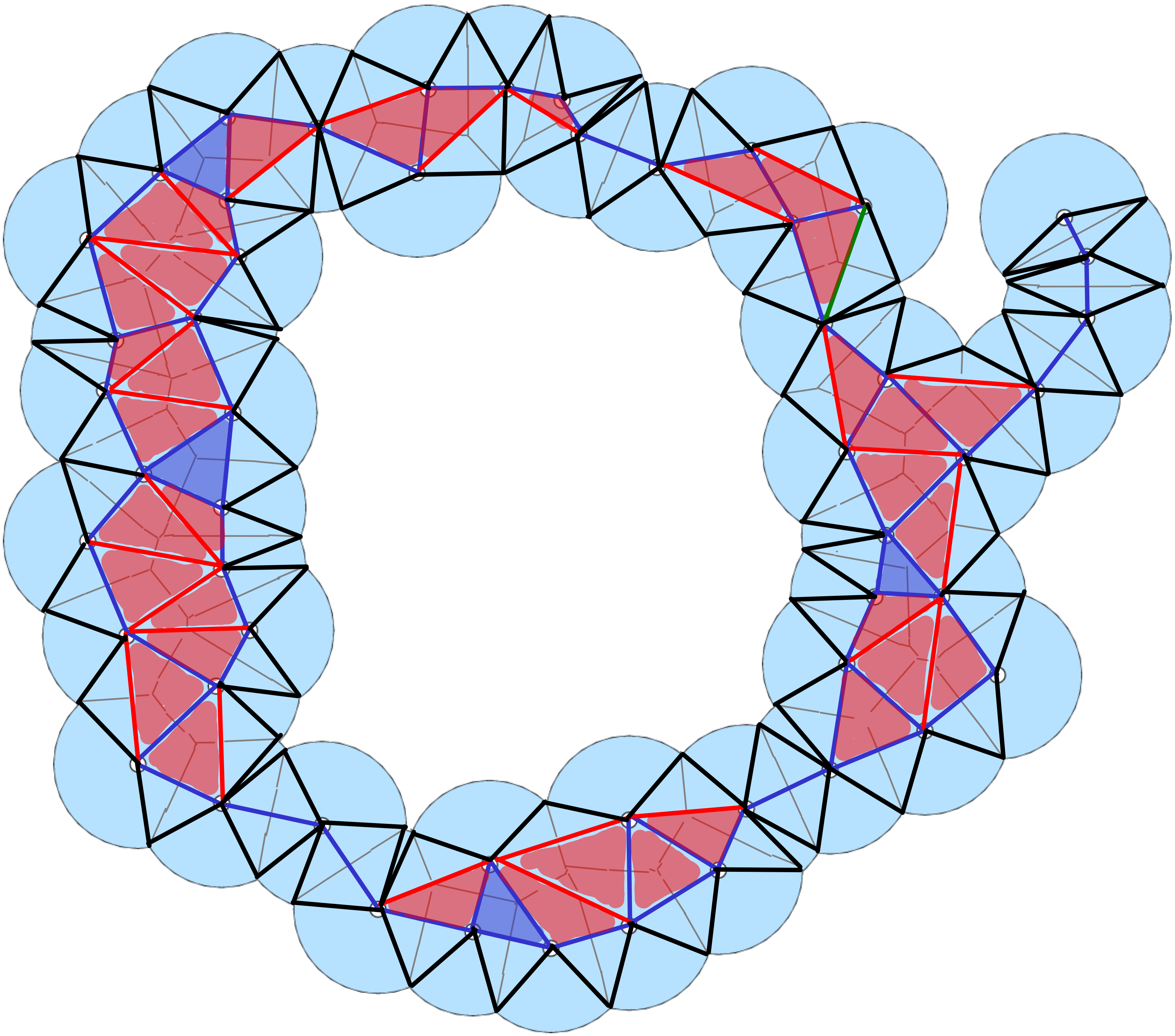
From Proteins to Simplicial Complexes

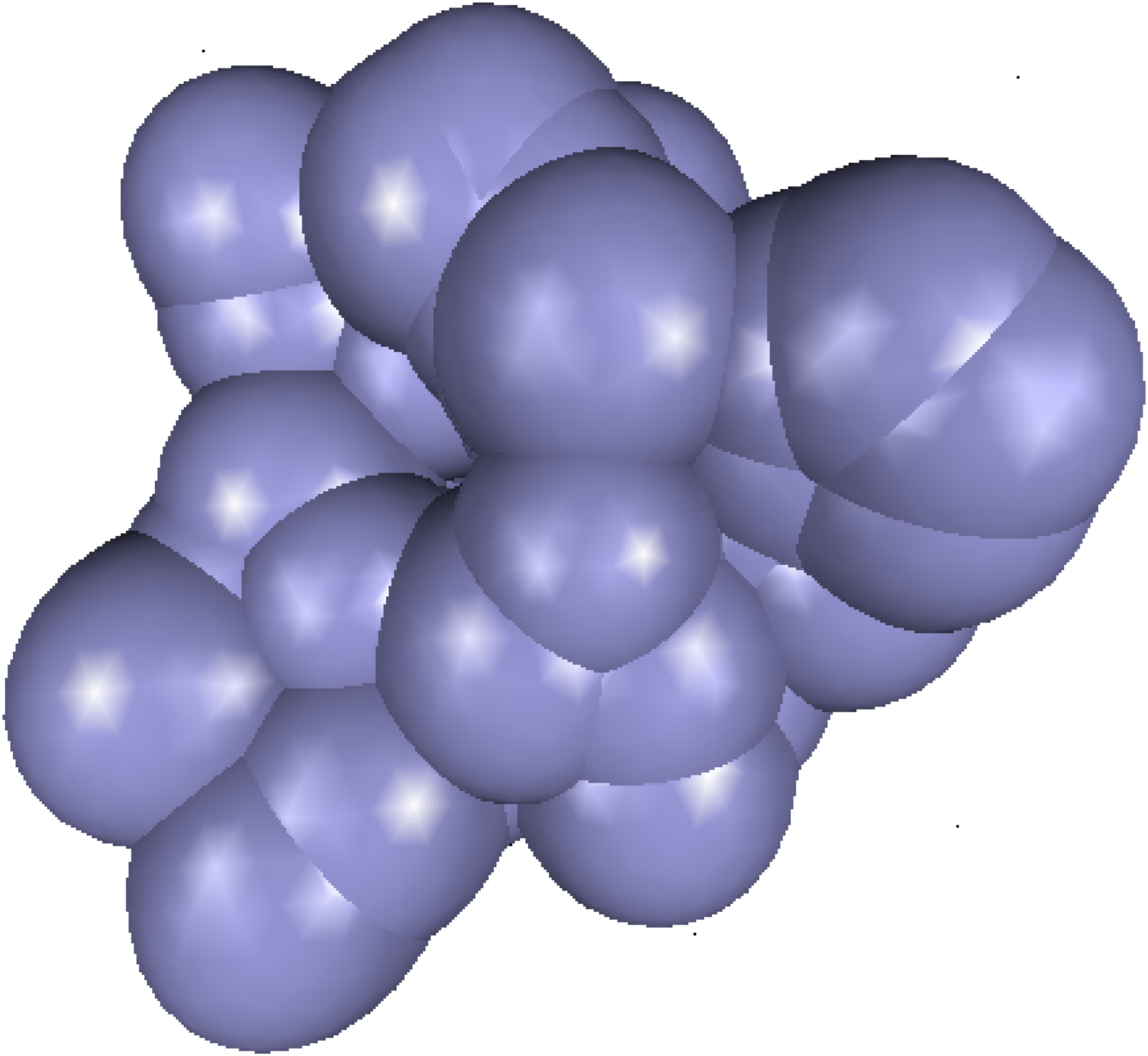


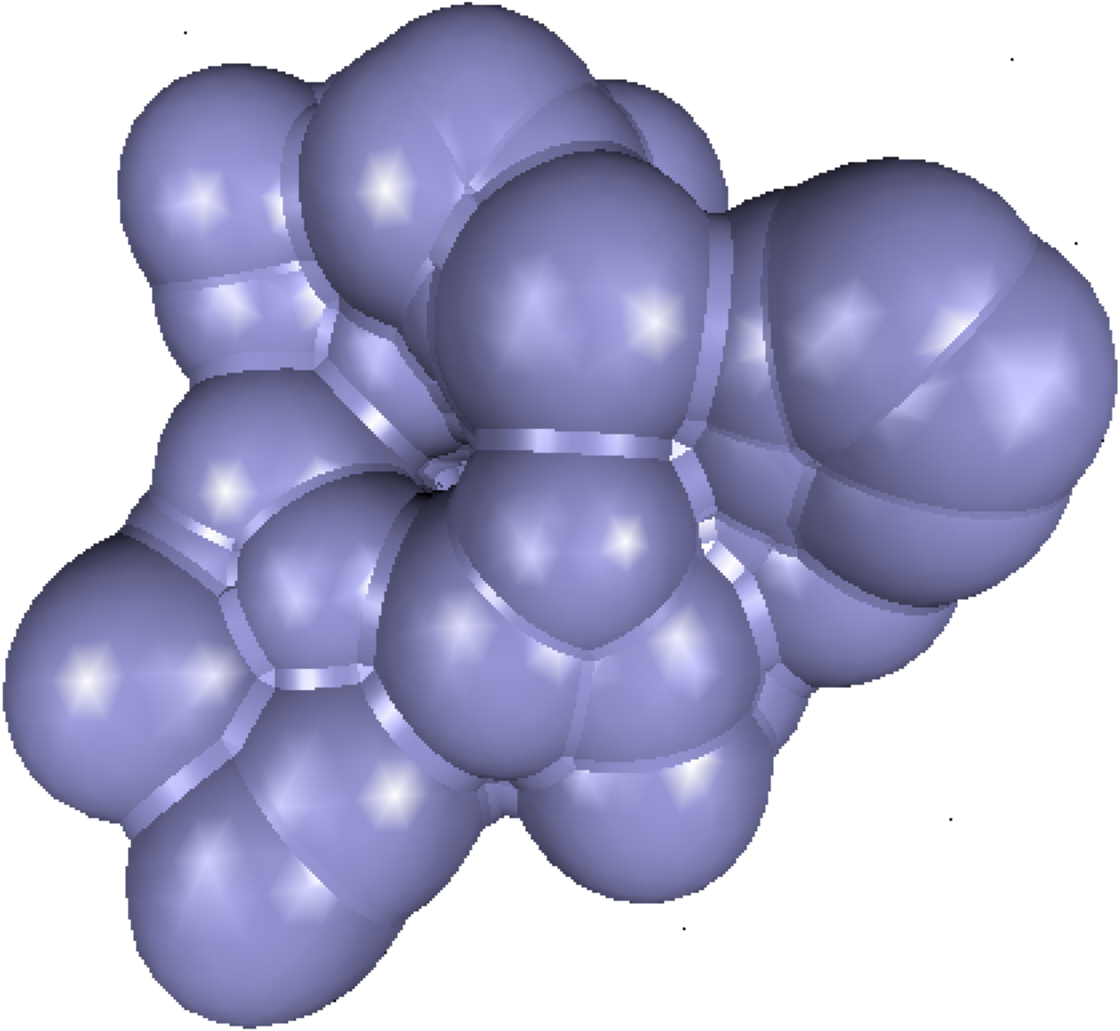
From Proteins to Simplicial Complexes

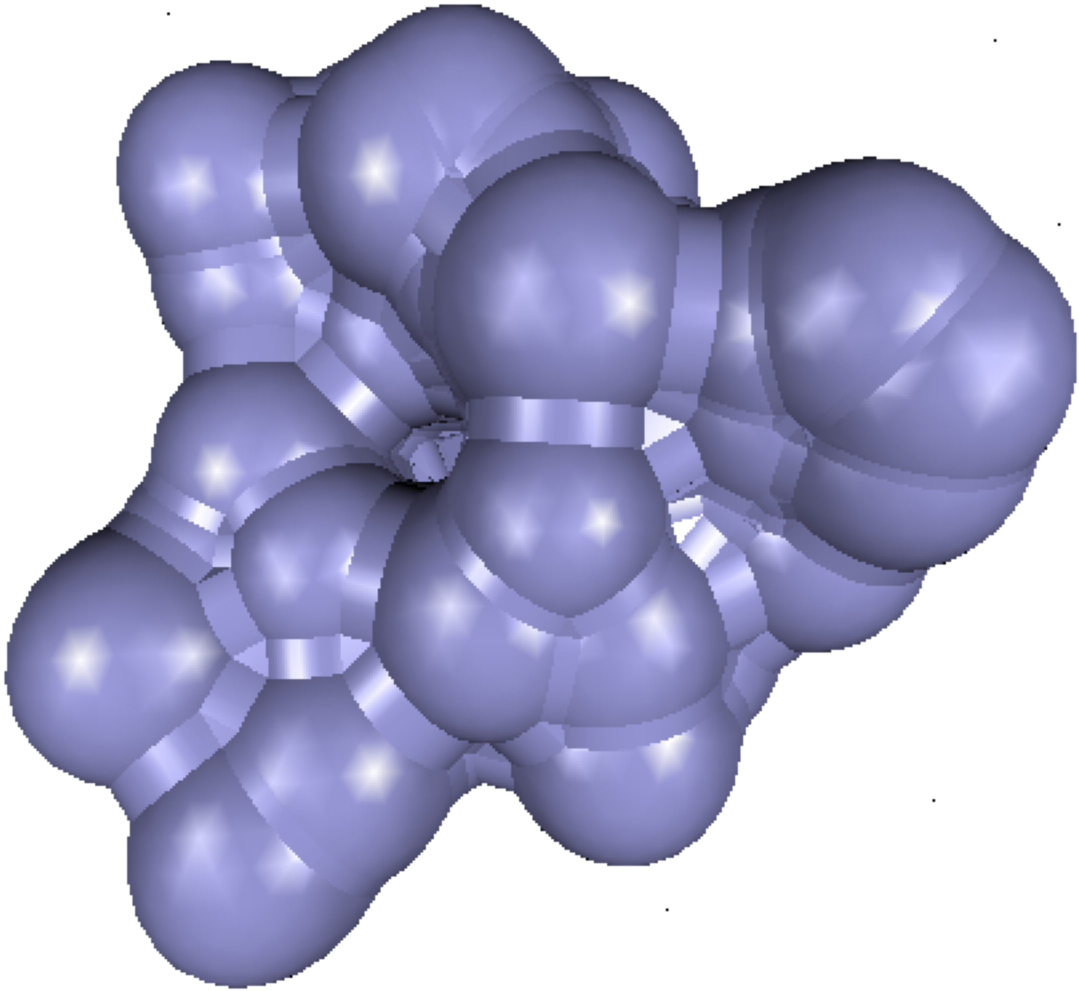


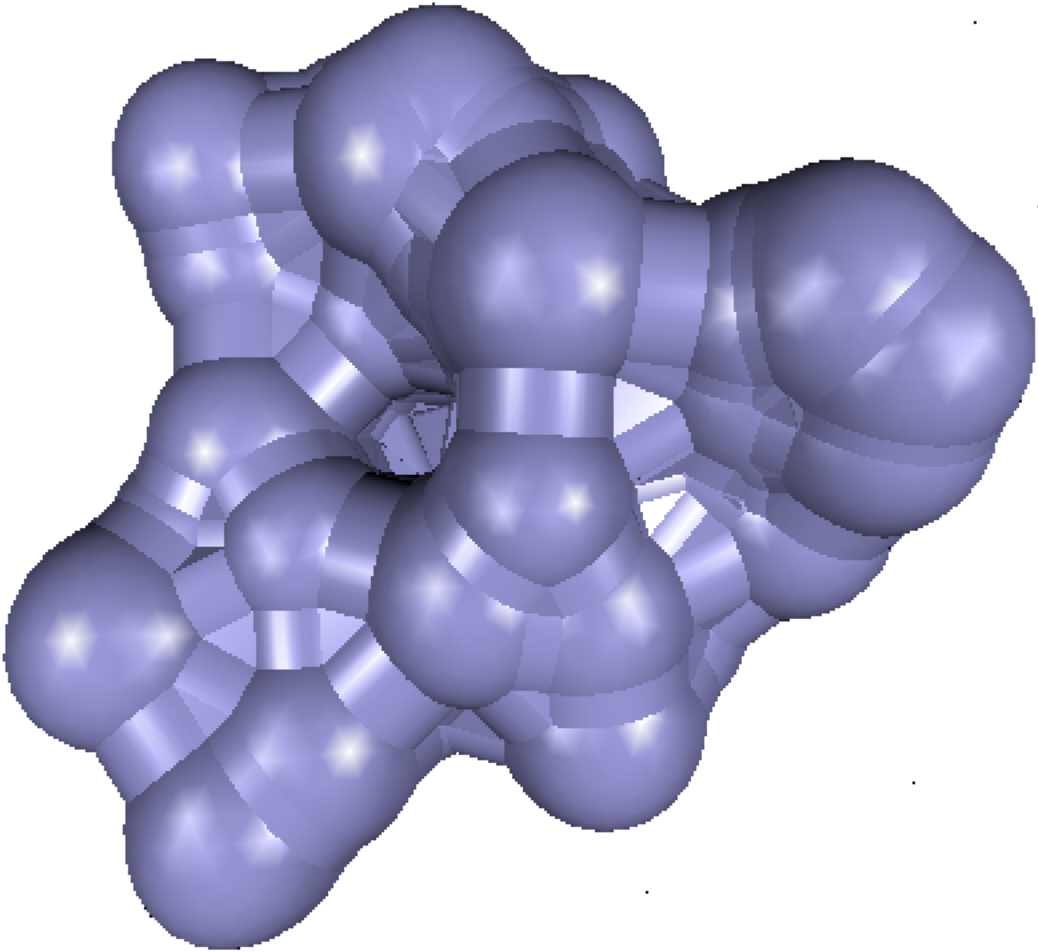
From Proteins to Simplicial Complexes

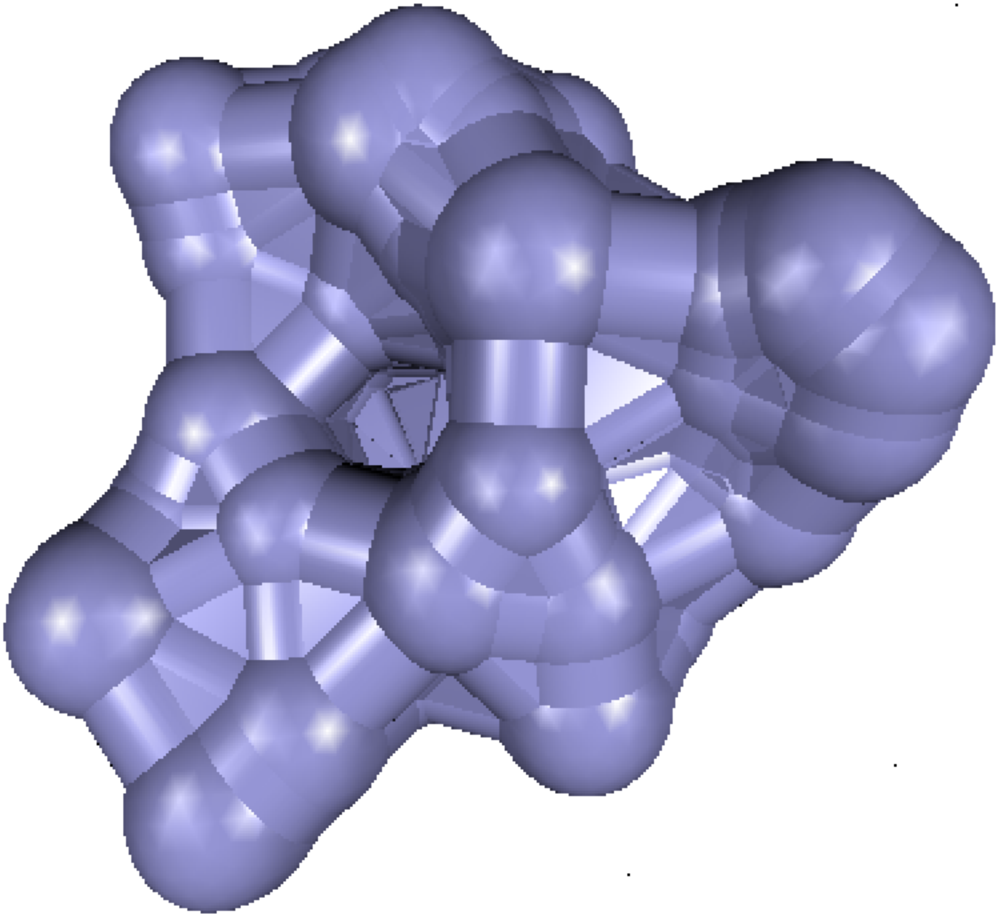


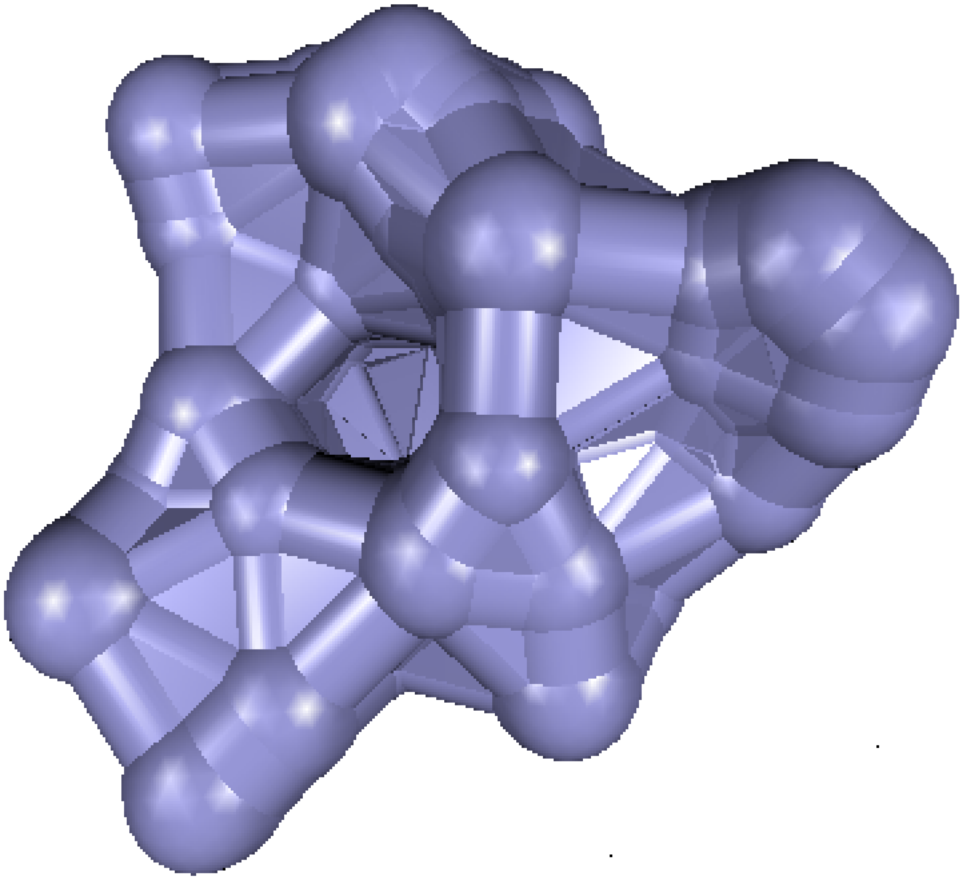


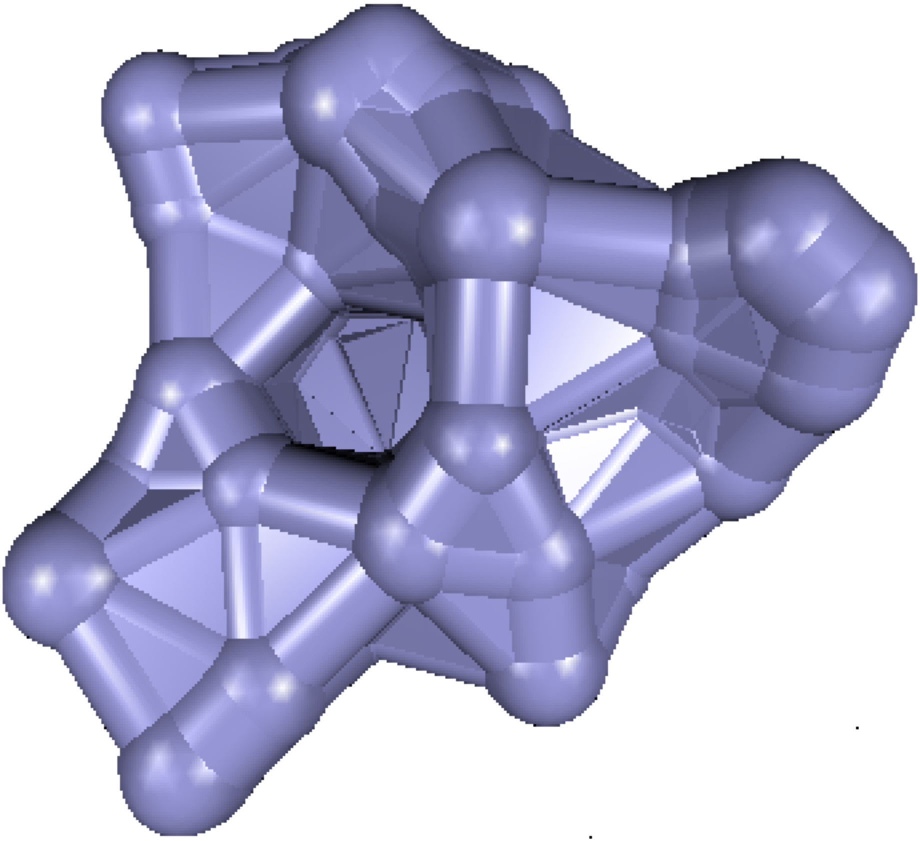


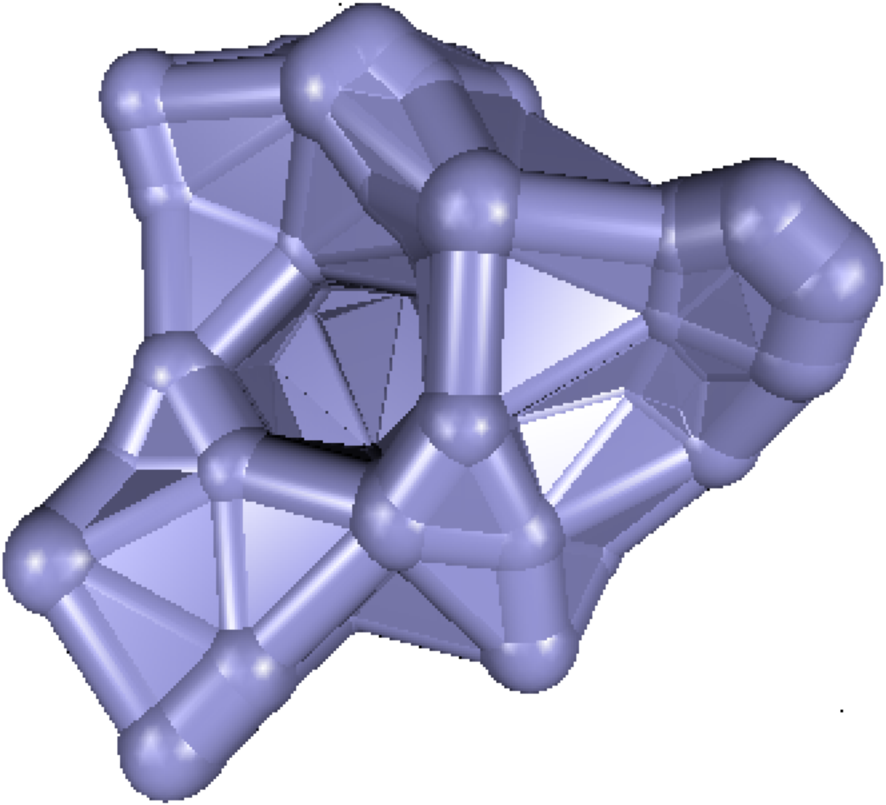


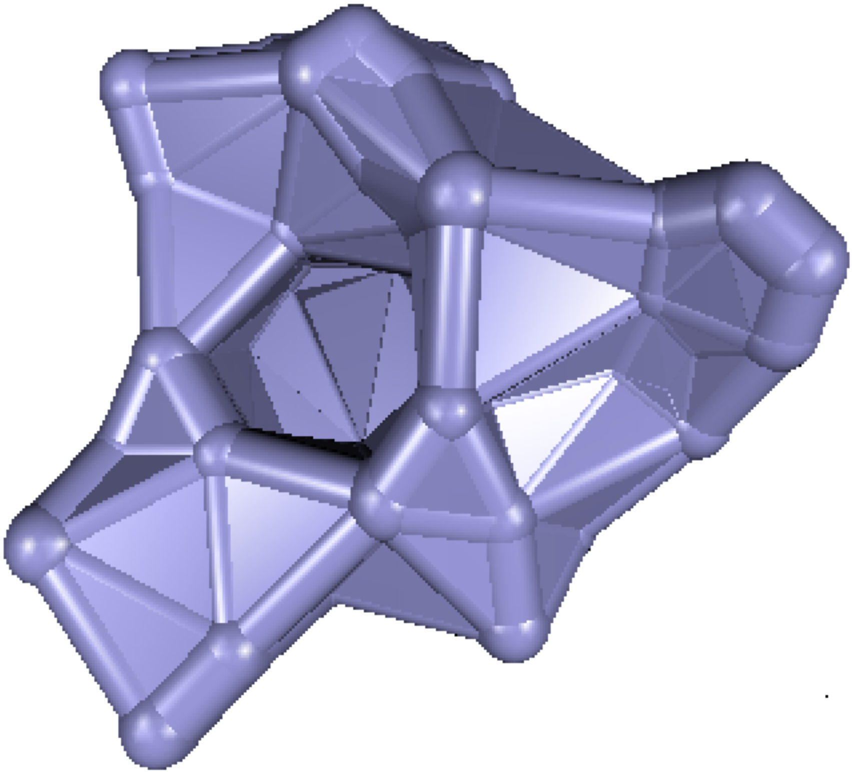


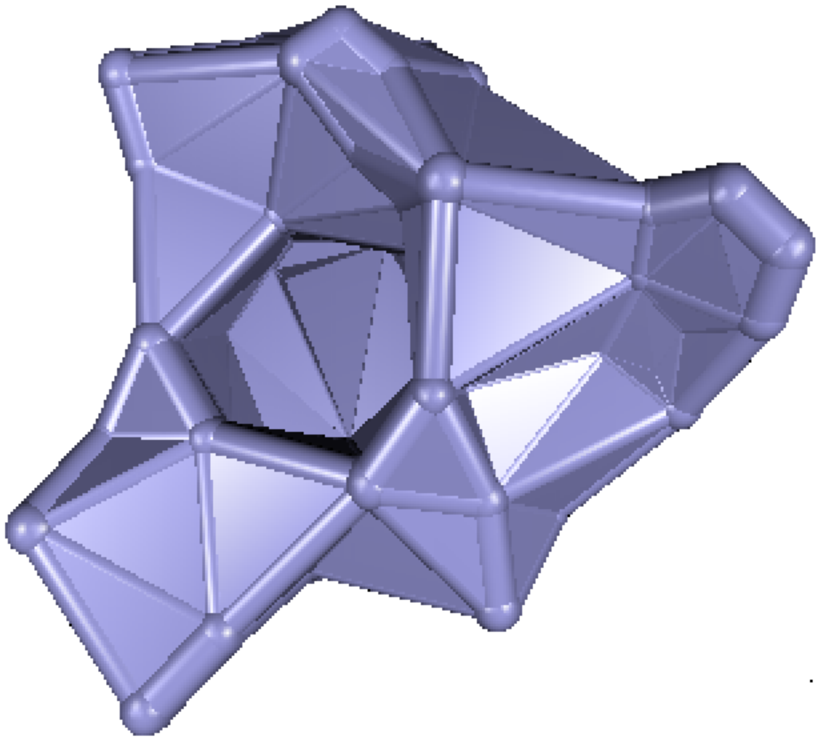


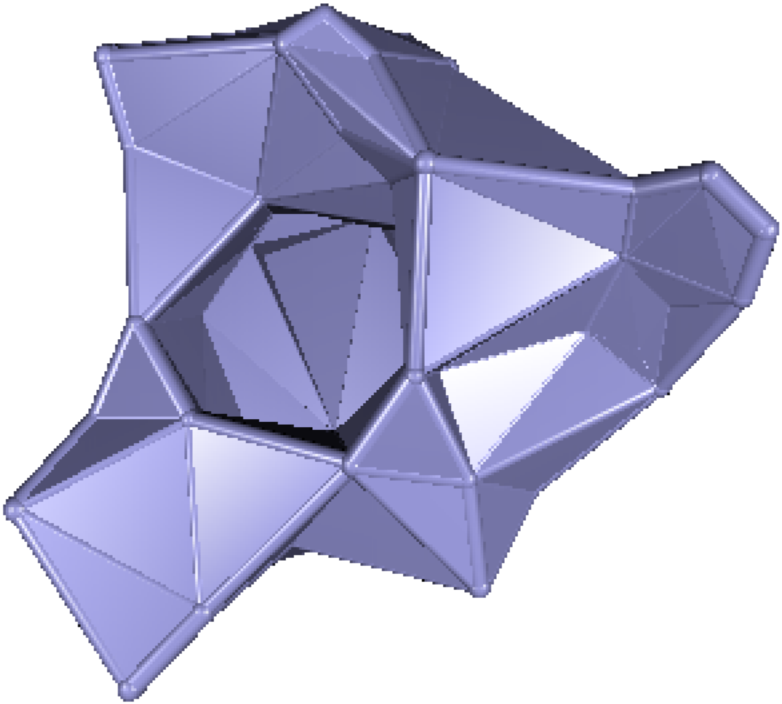


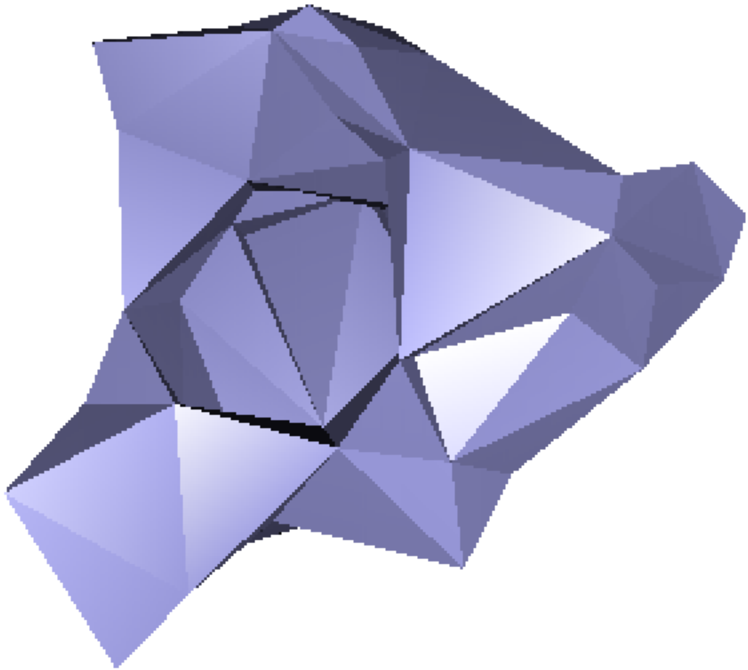












HISTORY

Alpha shapes $\mathbb{R}^2$:	E, Kirkpatrick, Seidel	1983
SoS:	E, Mücke	1990
Alpha shapes $\mathbb{R}^3$:	E, Mücke	1994

Shape, Homology, Persistence, and Stability

BETTI #s

in $\mathbb{R}^3$: β_0 = #components

β_1 = #tunnels

β_2 = #voids

Betti #s

in $\mathbb{R}^3$: β_0 = #components

β_1 = #tunnels

β_2 = #voids

VERTEX $\quad \bullet \quad \beta_0{+}{+}$

Betti #s

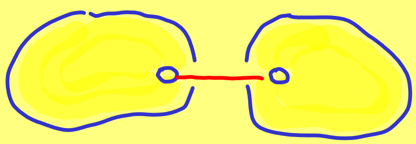
in $\mathbb{R}^3$: β_0 = #components

β_1 = #tunnels

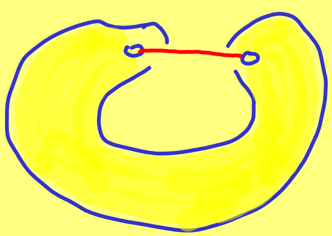
β_2 = #voids

VERTEX • β_0++

EDGE



β_0--



β_1++

Betti #s

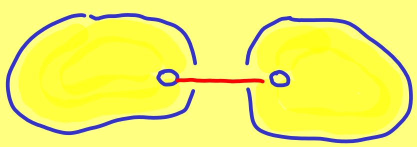
in $\mathbb{R}^3$: β_0 = #components

β_1 = #tunnels

β_2 = #voids

Vertex • $\beta_0{++}$

Edge

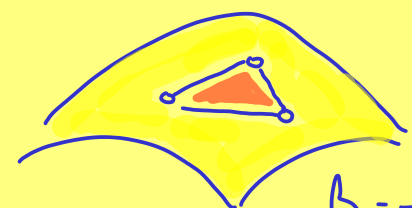


$\beta_0{--}$

$\beta_1{++}$

Triangle

$\beta_2{++}$



$\beta_1{--}$

Betti #s in $\mathbb{R}^3$:

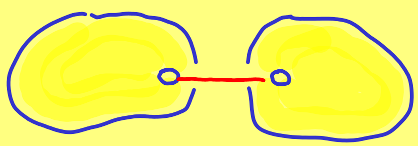
β_0 = #components

β_1 = #tunnels

β_2 = #voids

VERTEX • $\beta_0{++}$

EDGE

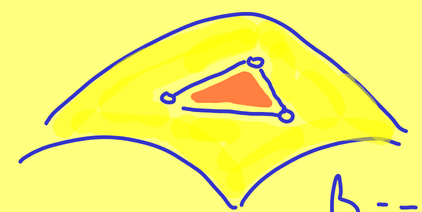


$\beta_0{--}$

$\beta_1{++}$

TRIANGLE

$\beta_2{++}$

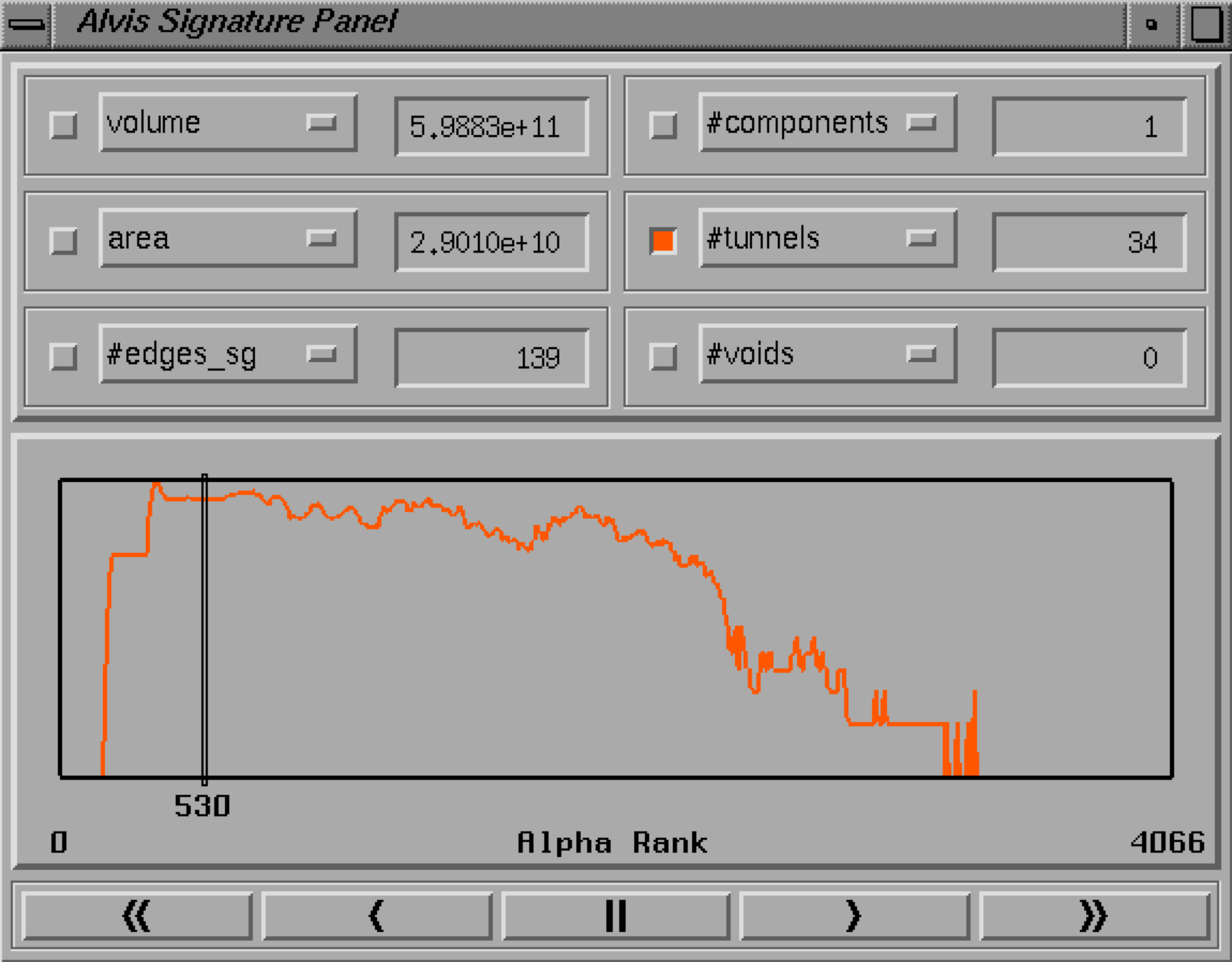


$\beta_1{--}$

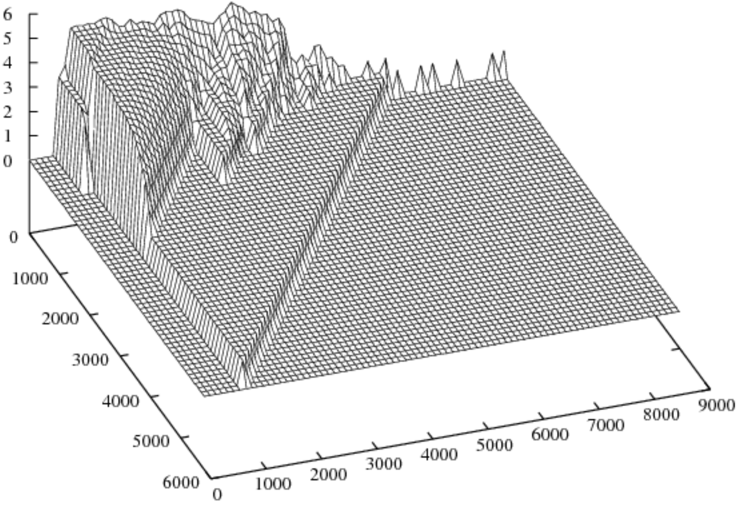
TETRAHEDRON $\beta_2{--}$

Number of Tunnels

r


Alvis Signature Panel
volume
5.9883e+11
#components
1
area
2.9010e+10
#tunnels
34
#edges_sg
139
#voids
0
530
0
Alpha Rank
4066

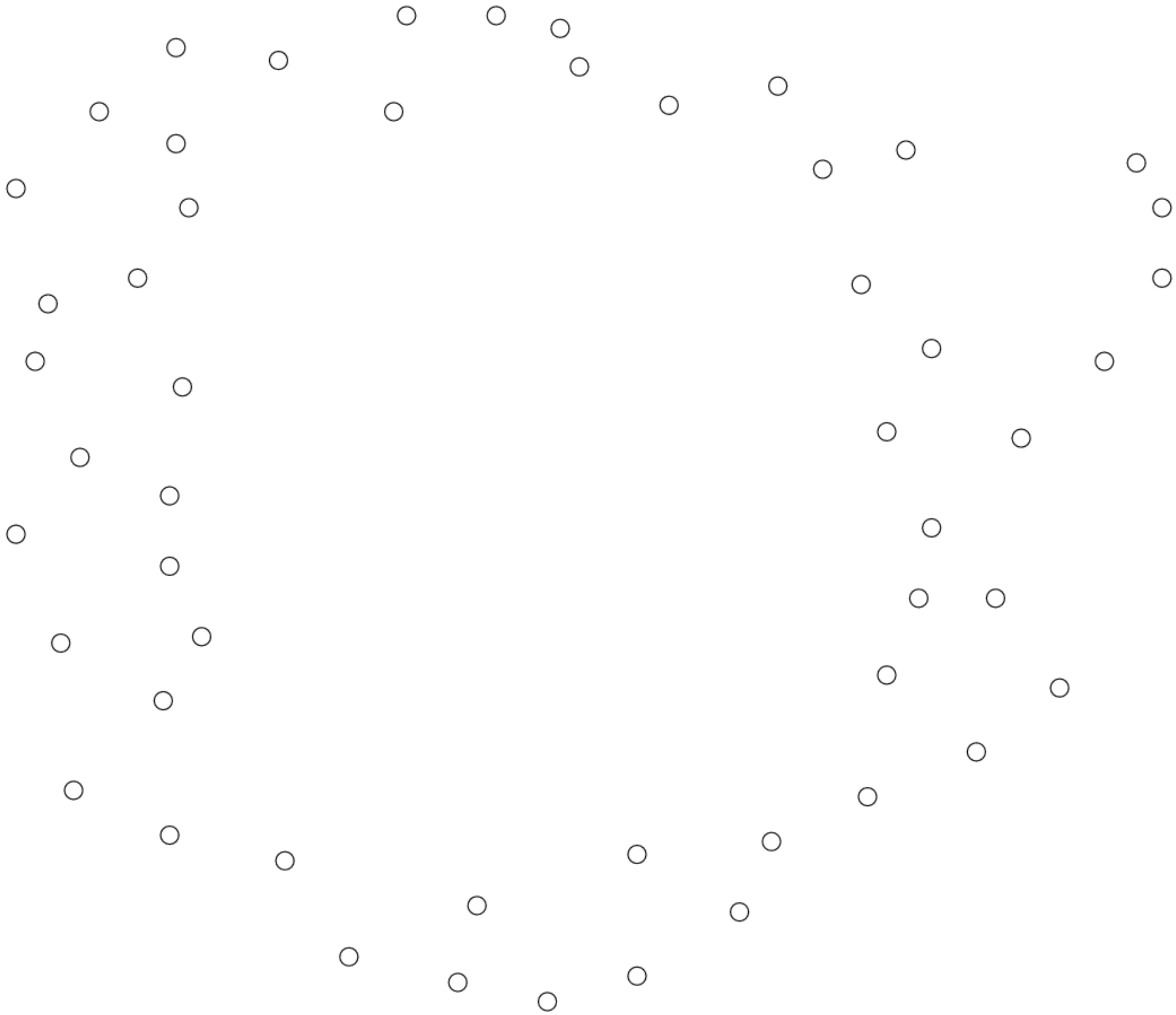
NUMBER OF TUNNELS

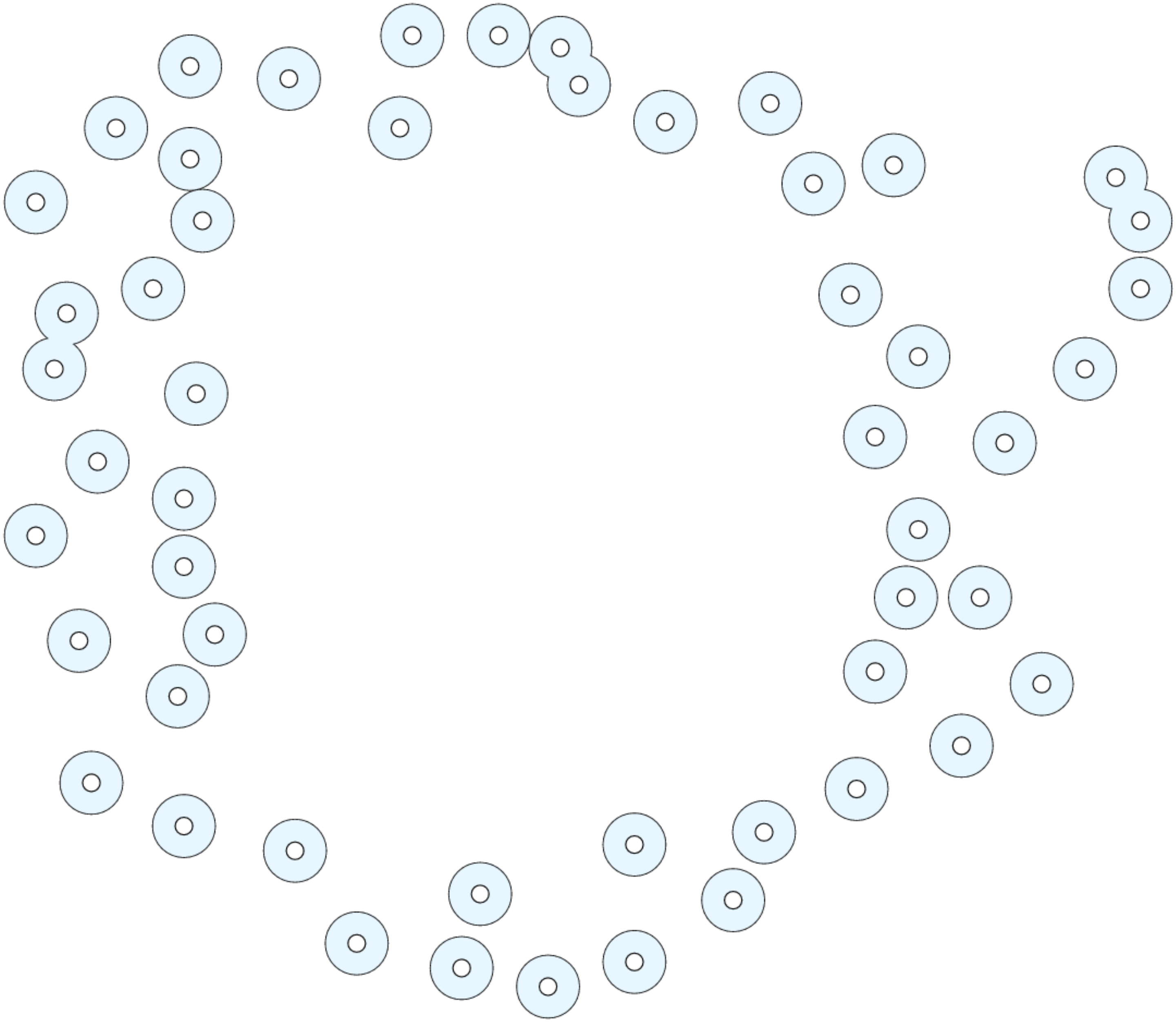


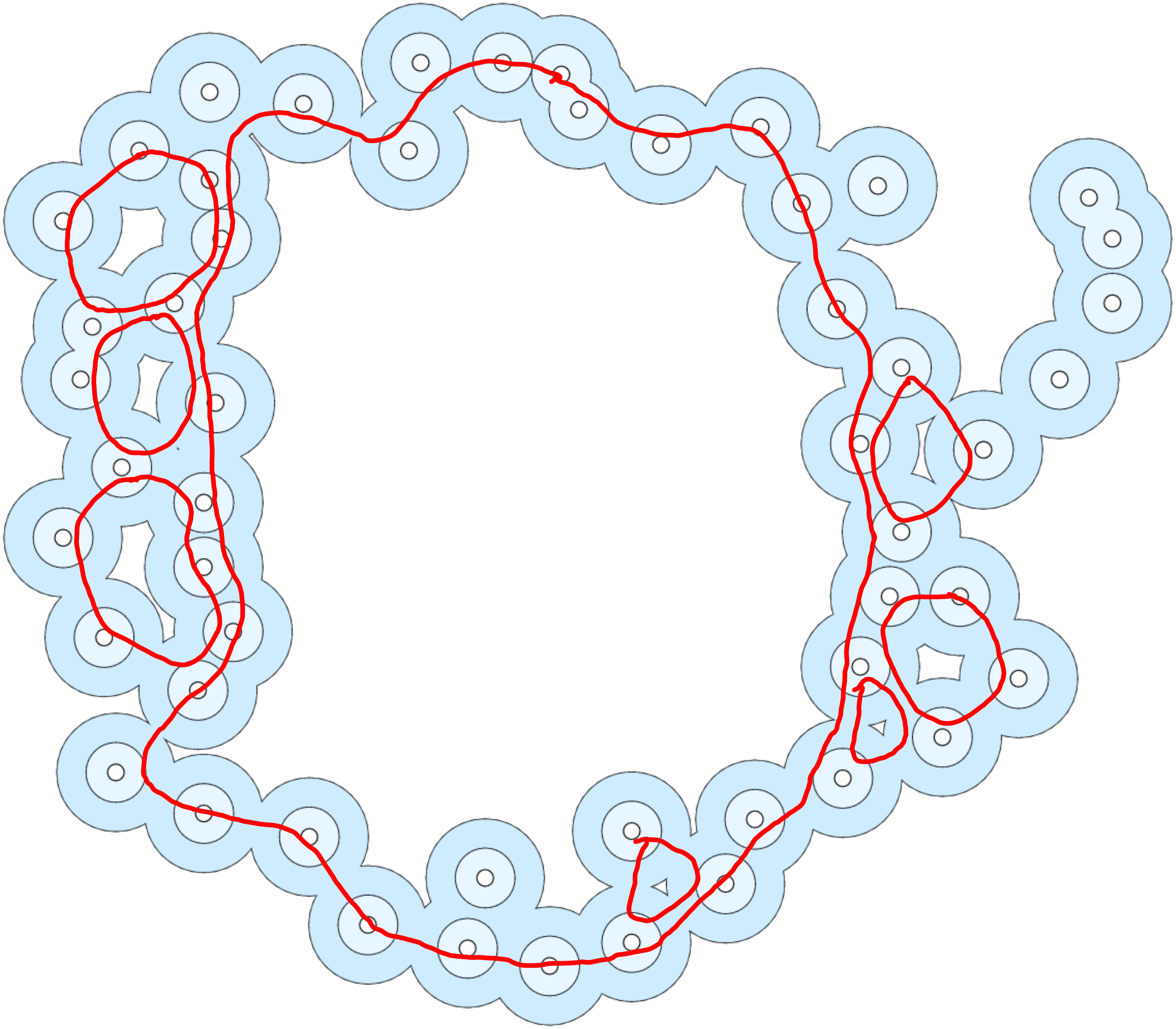
History

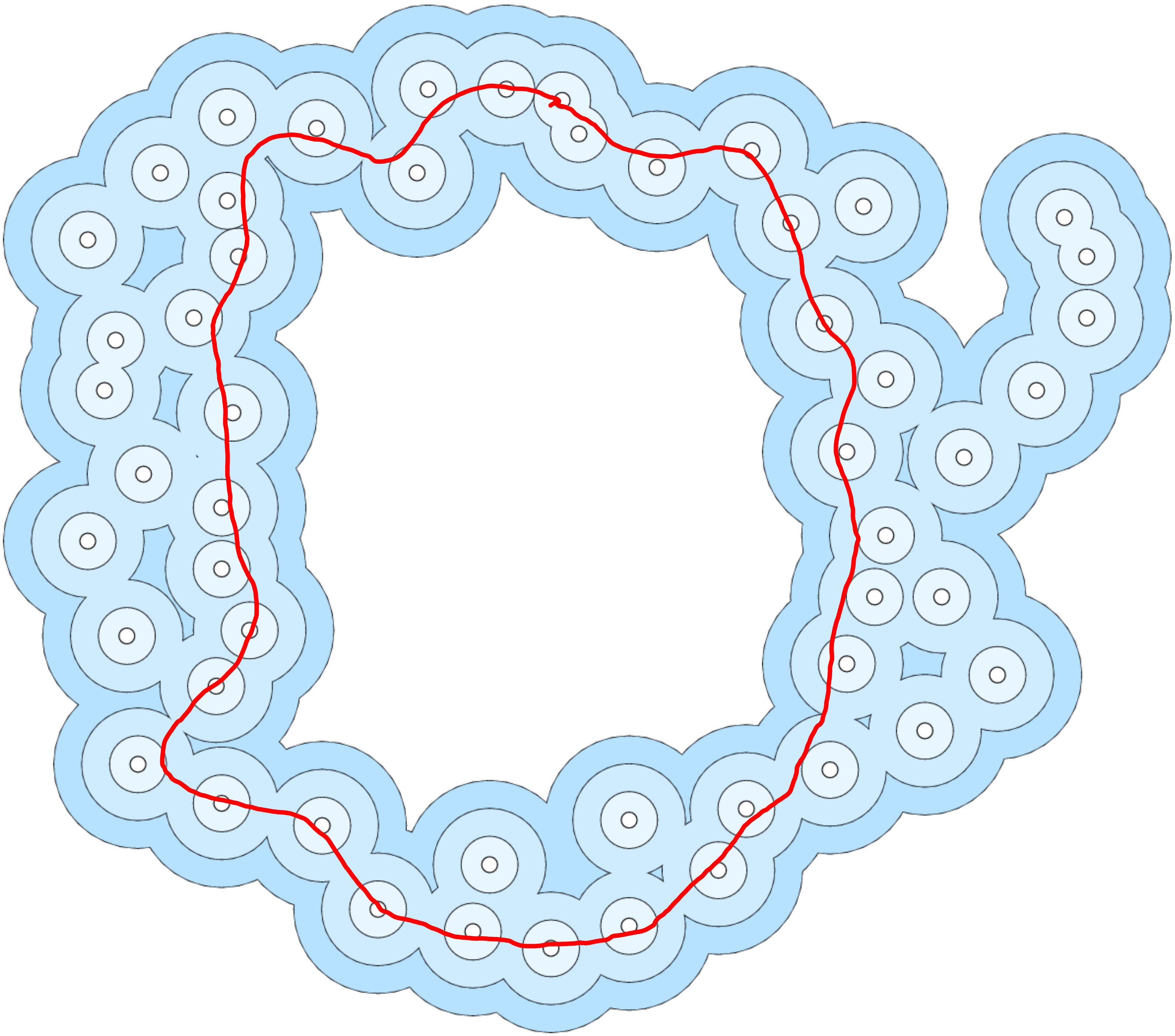
Incremental algorithm: Delfinado, E 1995

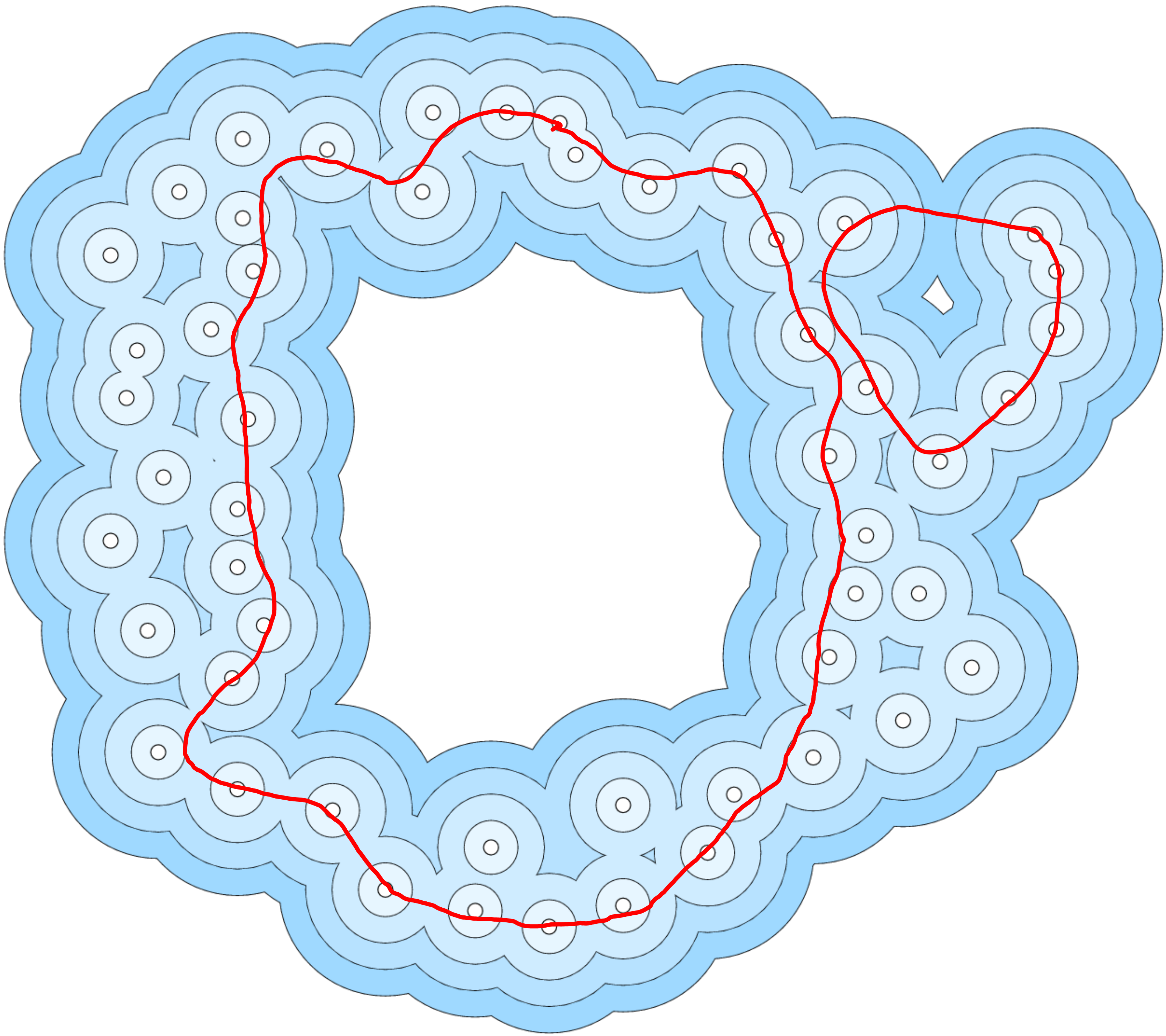
Shape, Homology, Persistence, and Stability

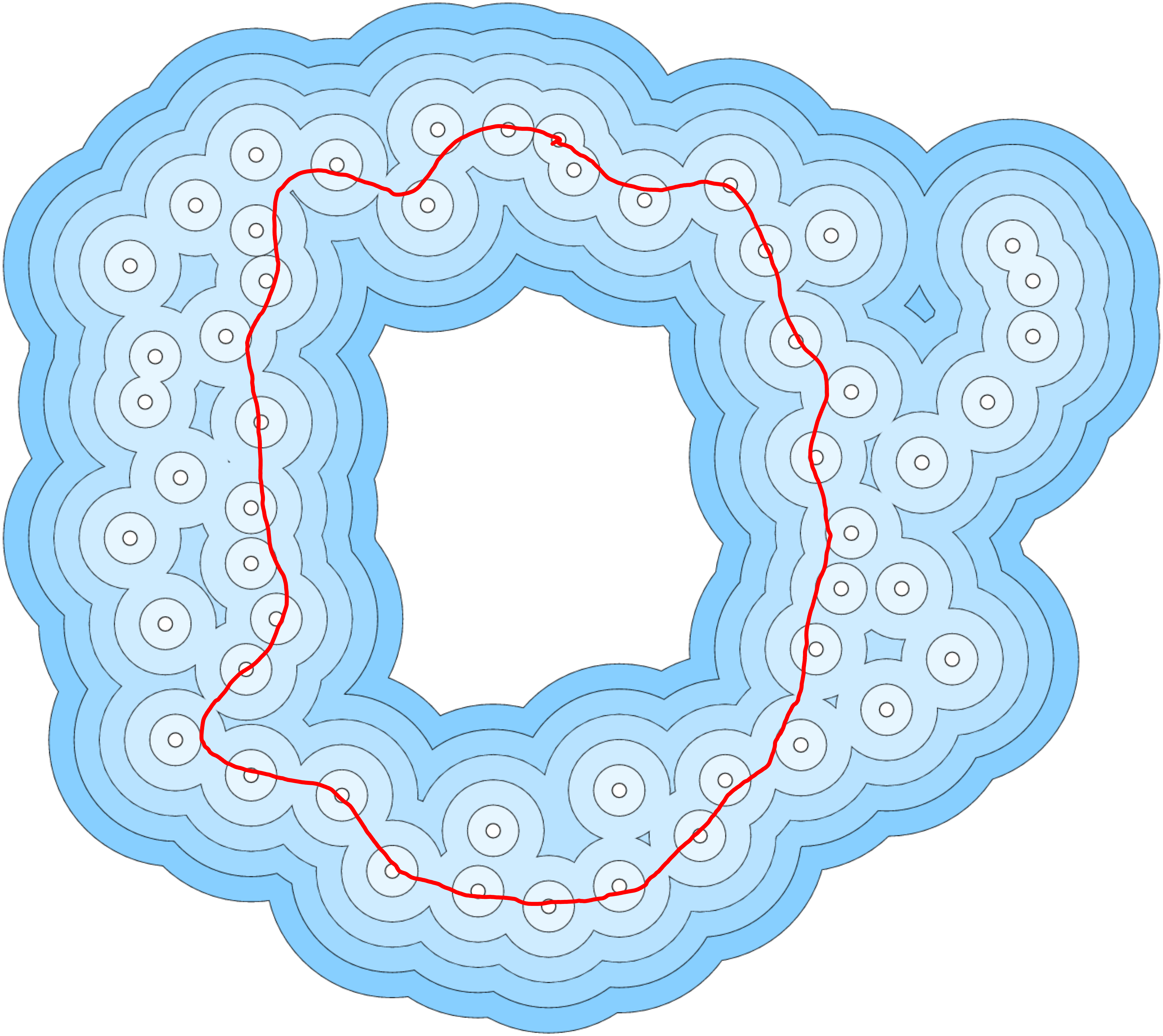


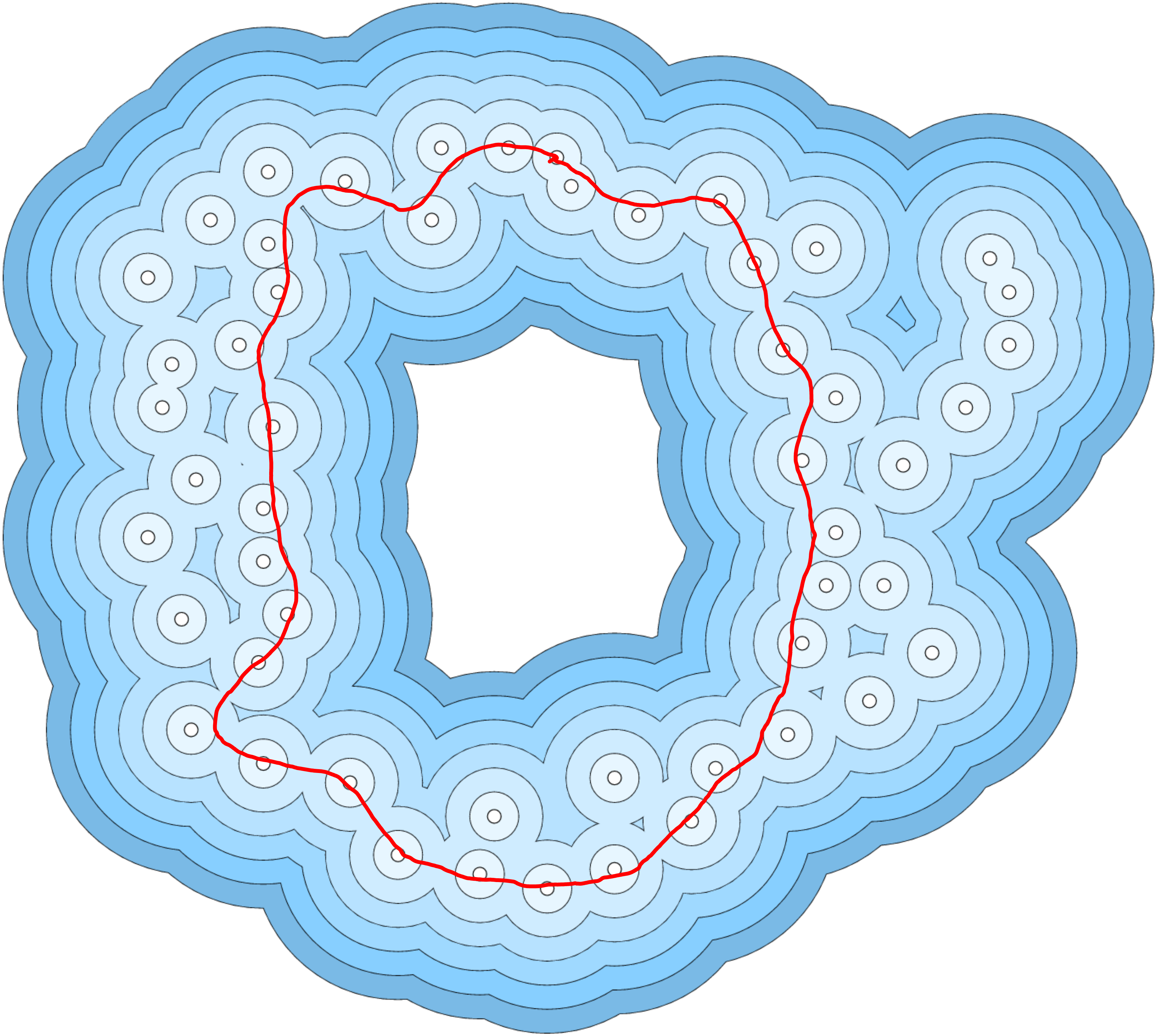


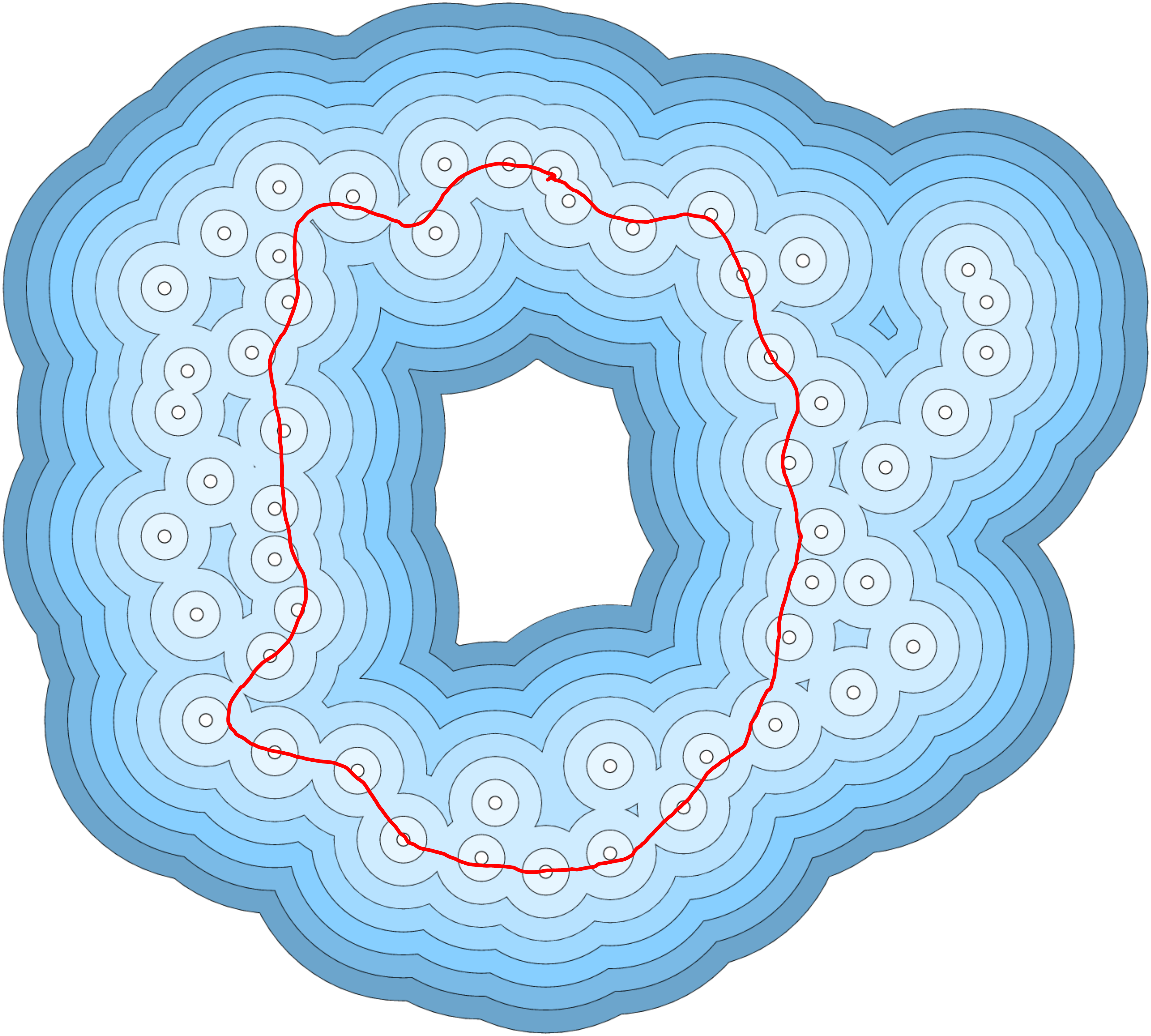


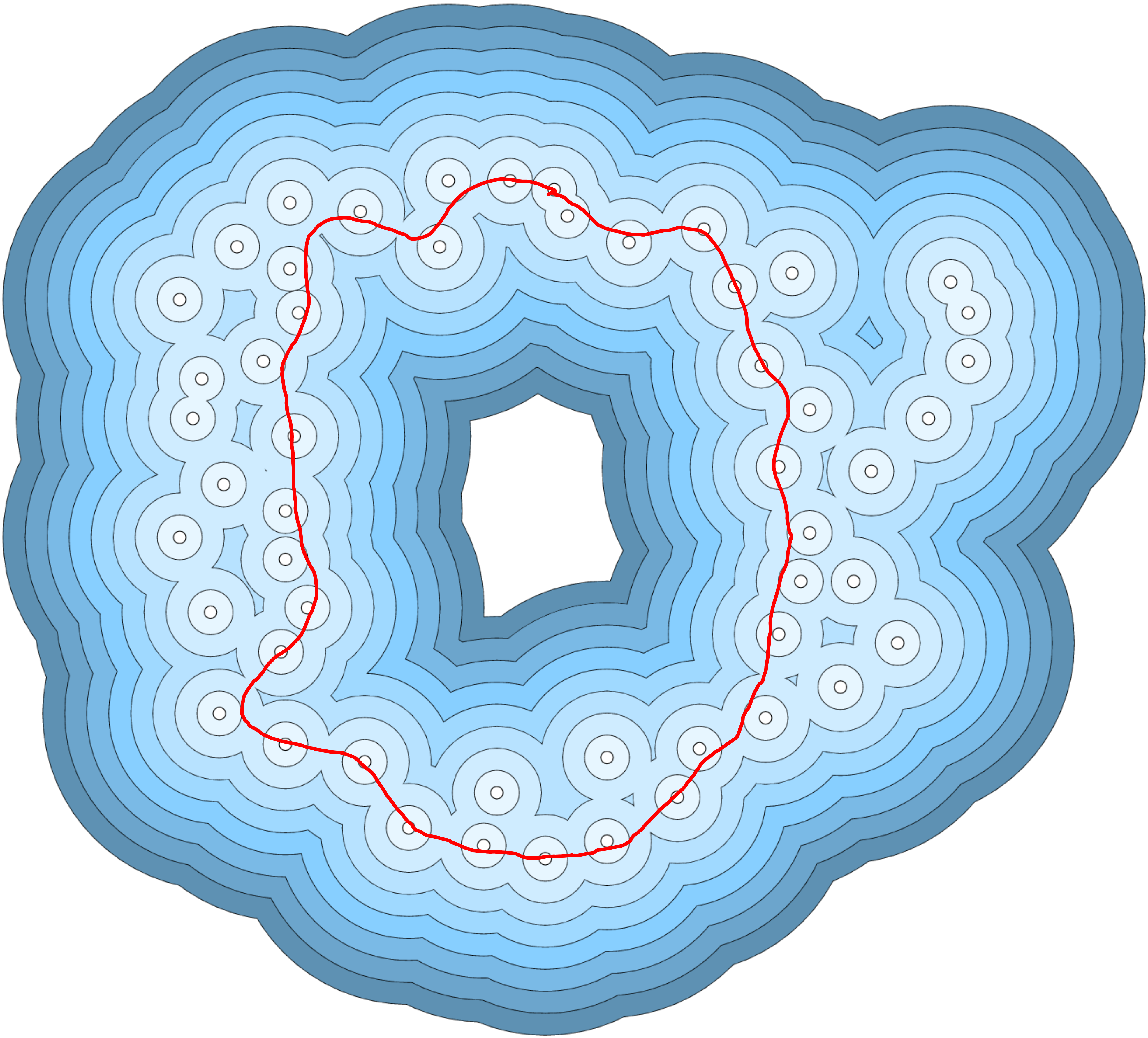








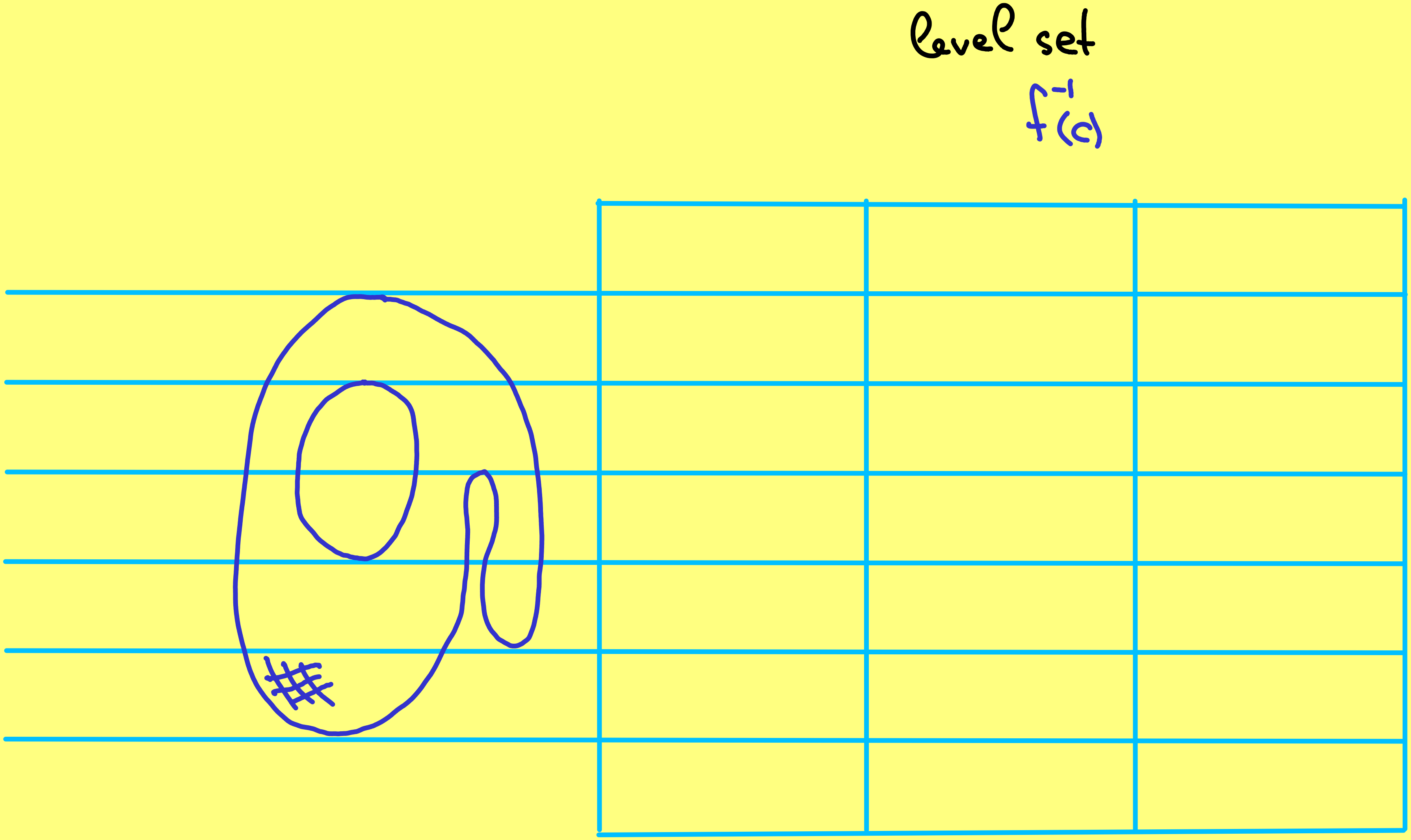




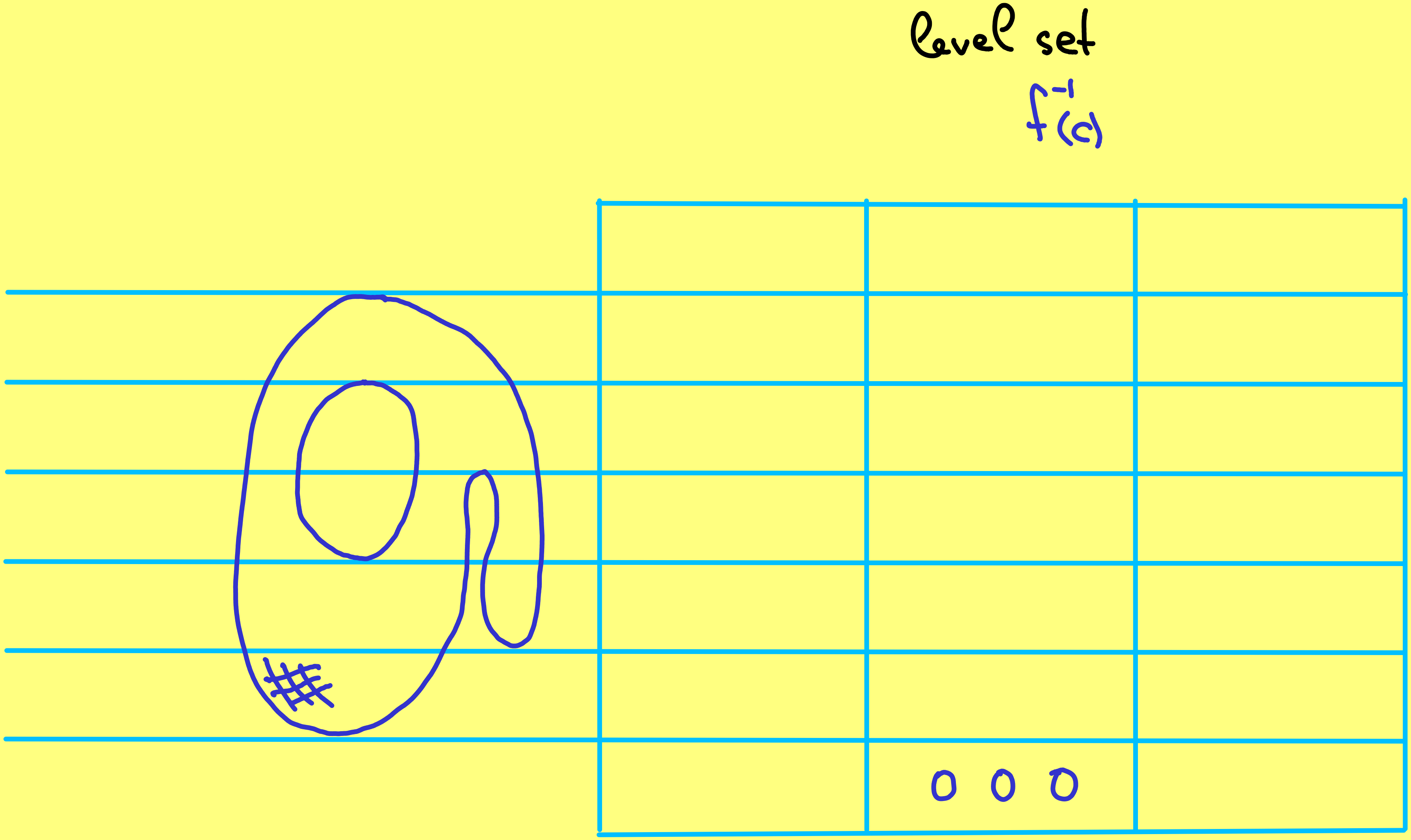
Birth & Death

Birth & Death

Birth & Death



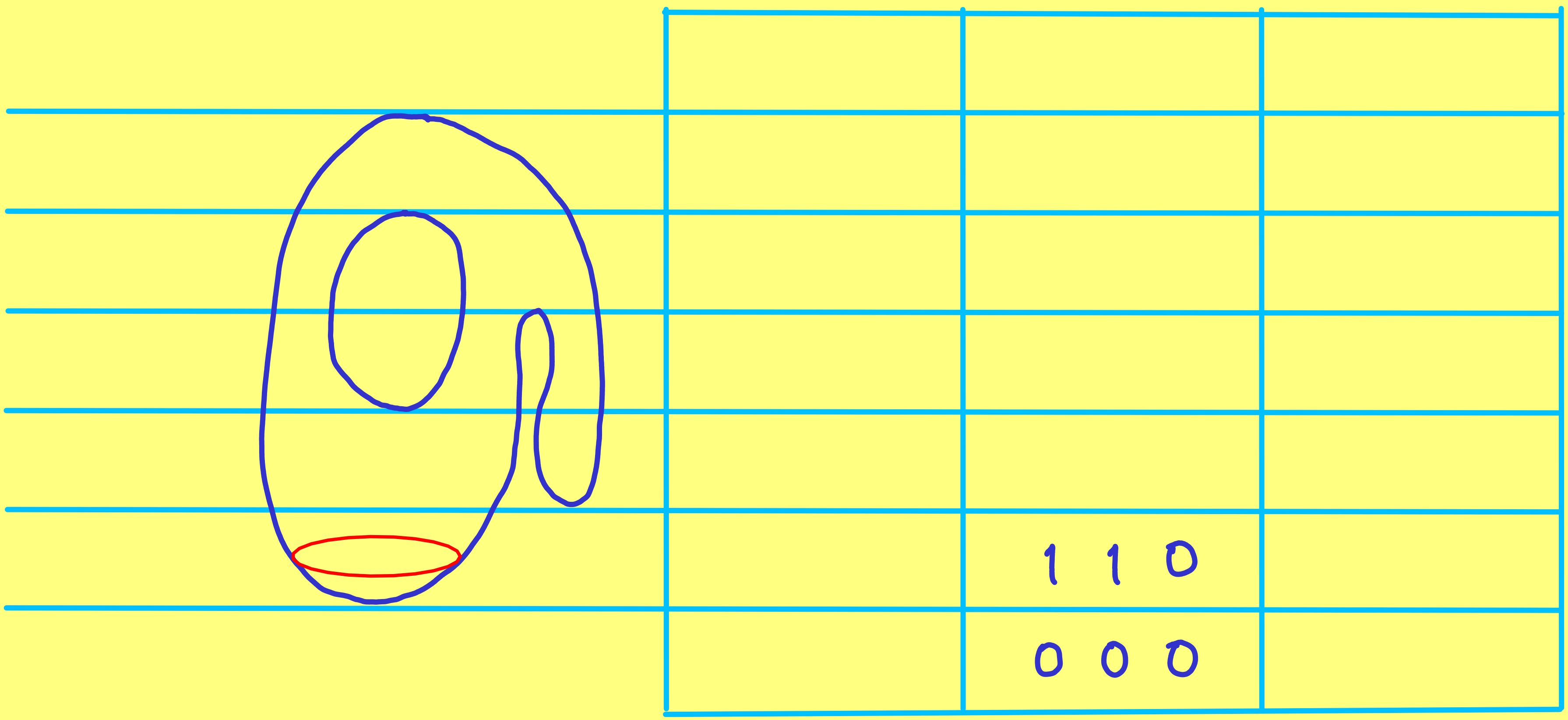
Birth & Death



Birth & Death

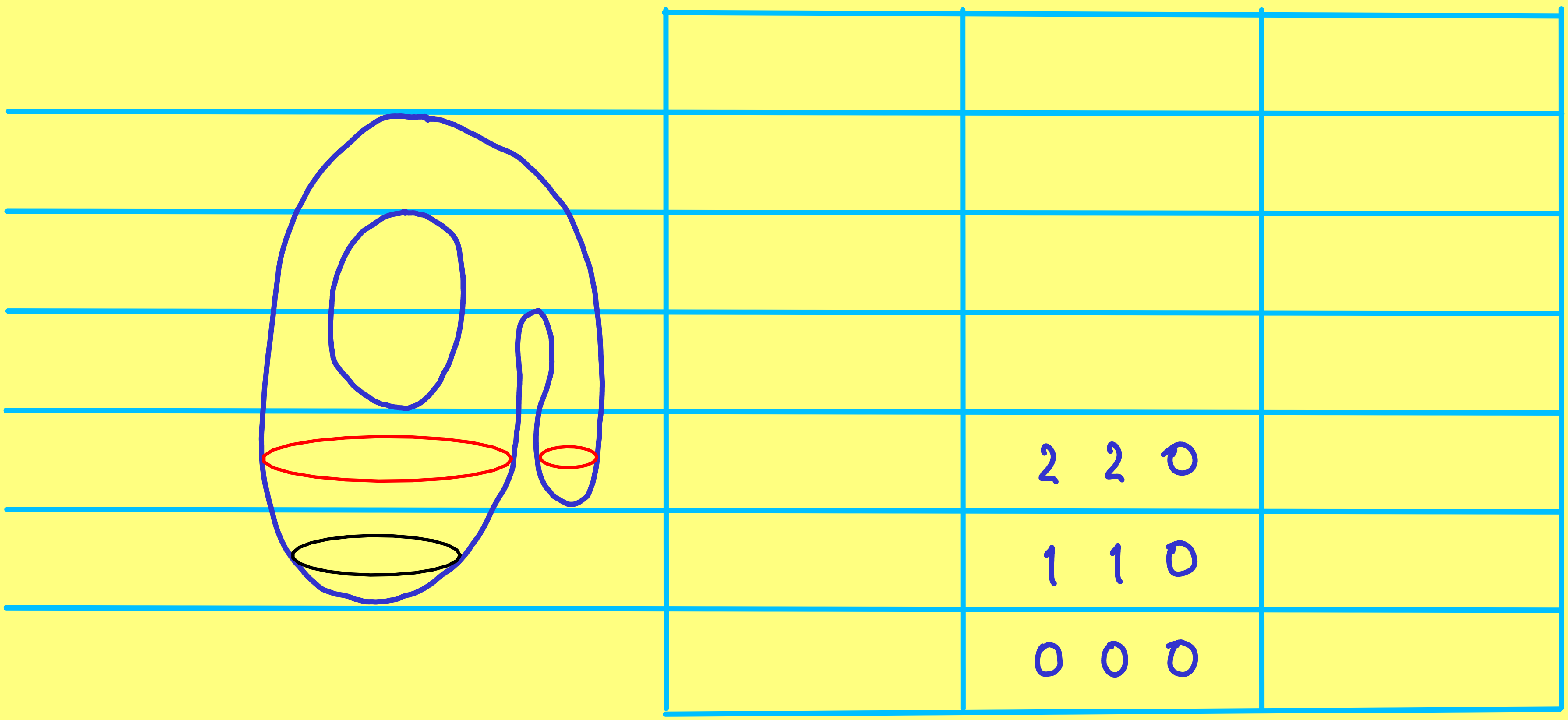
level set

$f^{-1}(c)$



Birth & Death

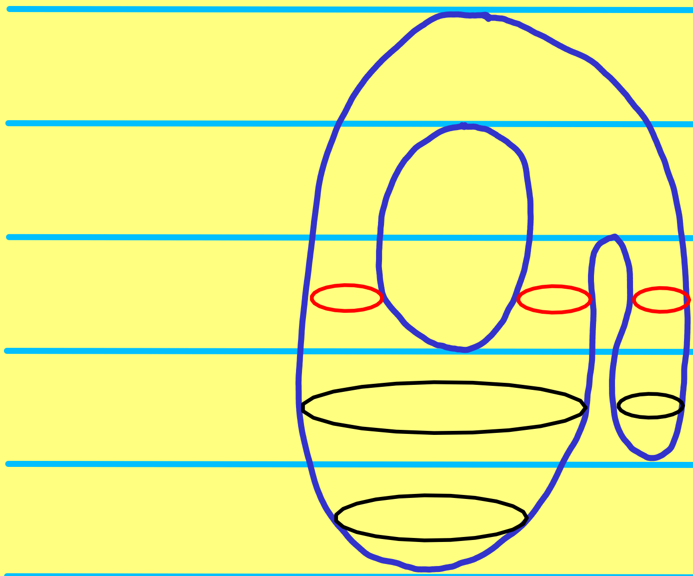
level set $f^{-1}(c)$



Birth & Death

level set

$f^{-1}(c)$

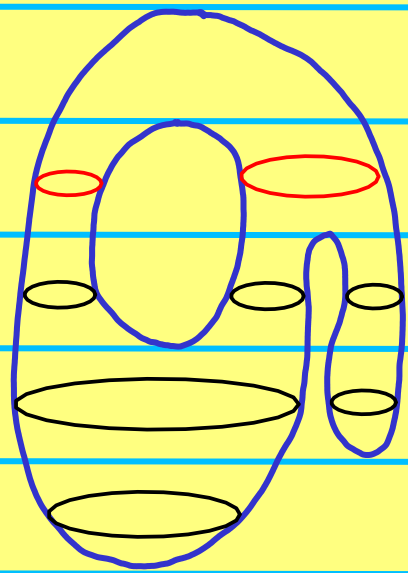


	3 3 0	
	2 2 0	
	1 1 0	
	0 0 0	

Birth & Death

level set

$f^{-1}(c)$

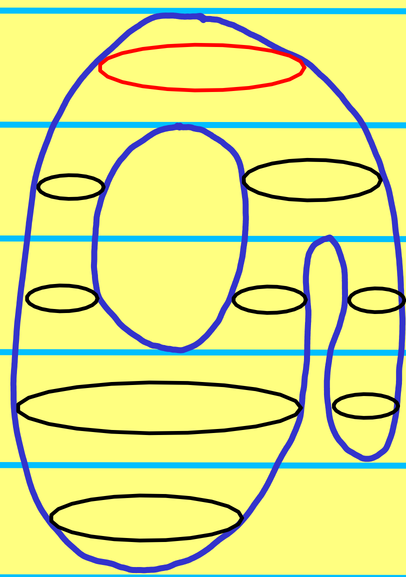


	2 2 0	
	3 3 0	
	2 2 0	
	1 1 0	
	0 0 0	

Birth & Death

level set

$f^{-1}(c)$

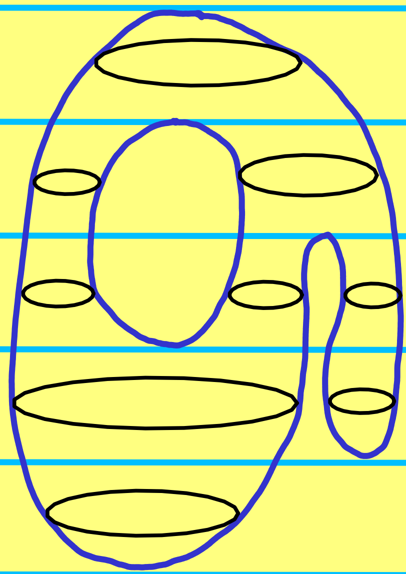


	1 1 0	
	2 2 0	
	3 3 0	
	2 2 0	
	1 1 0	
	0 0 0	

Birth & Death

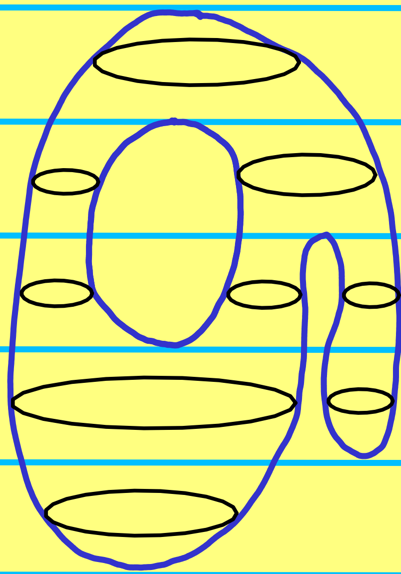
level set

$f^{-1}(c)$



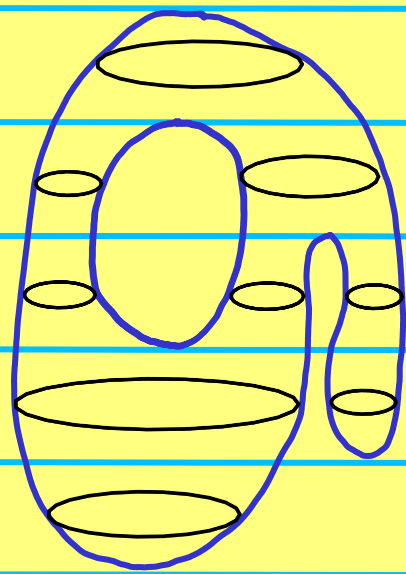
	0 0 0	
	1 1 0	
	2 2 0	
	3 3 0	
	2 2 0	
	1 1 0	
	0 0 0	

Birth & Death



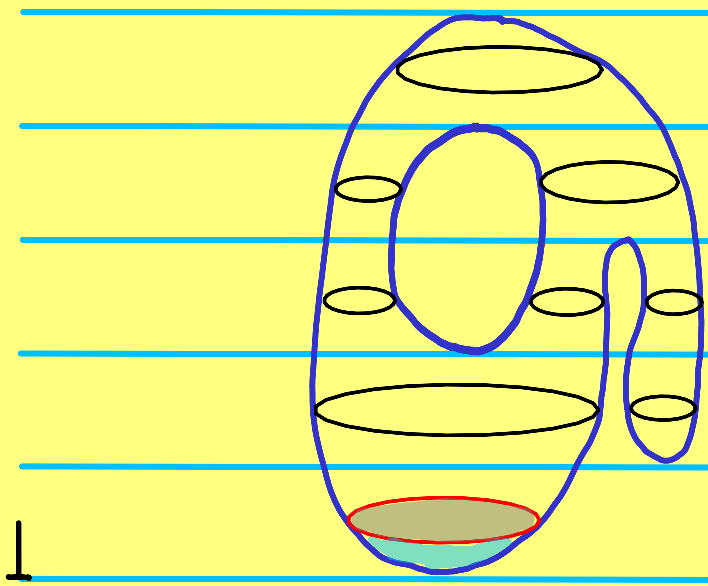
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	
	0 0 0	
	1 1 0	
	2 2 0	
	3 3 0	
	2 2 0	
	1 1 0	
	0 0 0	

Birth & Death



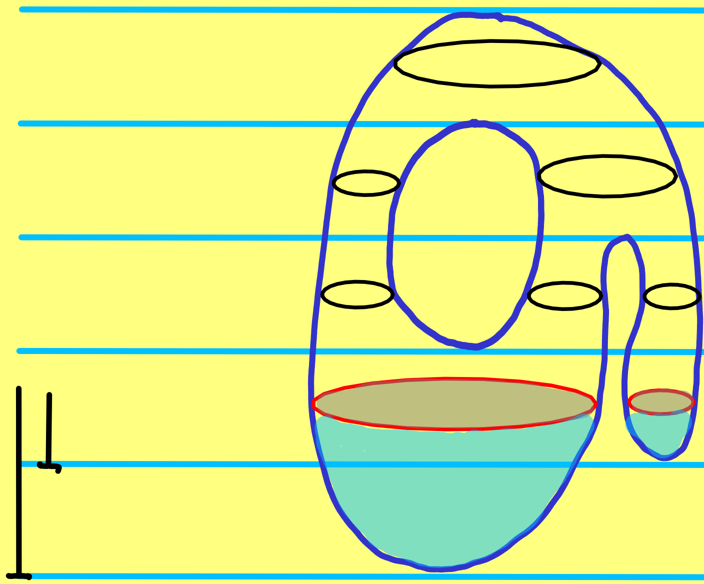
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	
	0 0 0	
	1 1 0	
	2 2 0	
	3 3 0	
	2 2 0	
	1 1 0	
0 0 0	0 0 0	

Birth & Death



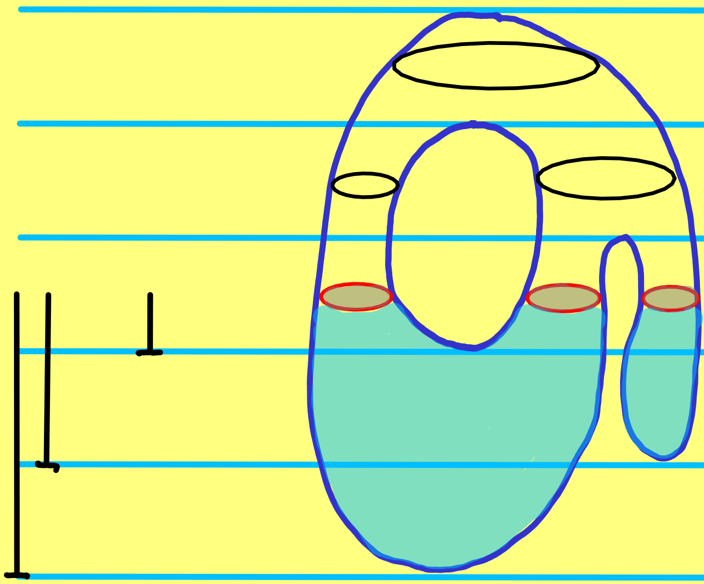
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	
	0 0 0	
	1 1 0	
	2 2 0	
	3 3 0	
	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



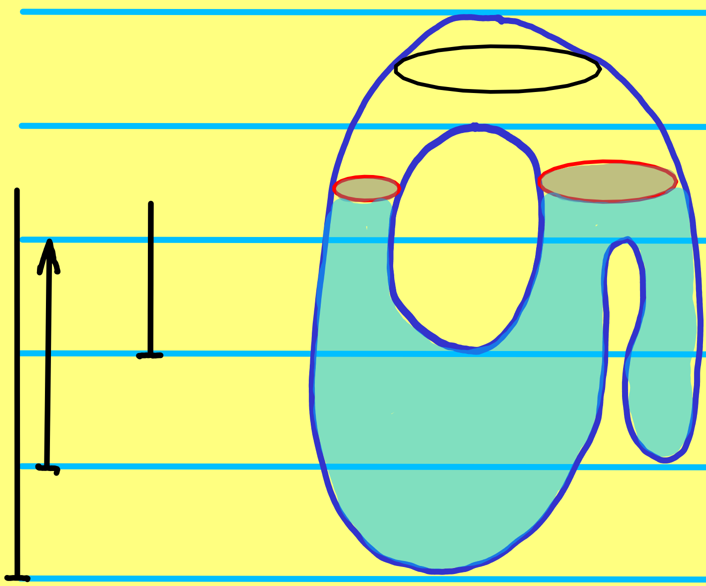
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	
	0 0 0	
	1 1 0	
	2 2 0	
	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



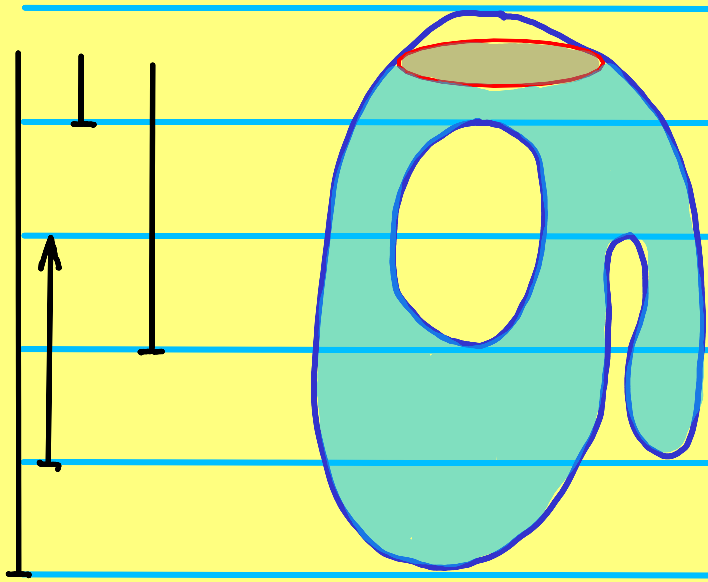
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	
	0 0 0	
	1 1 0	
	2 2 0	
2 1 0	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



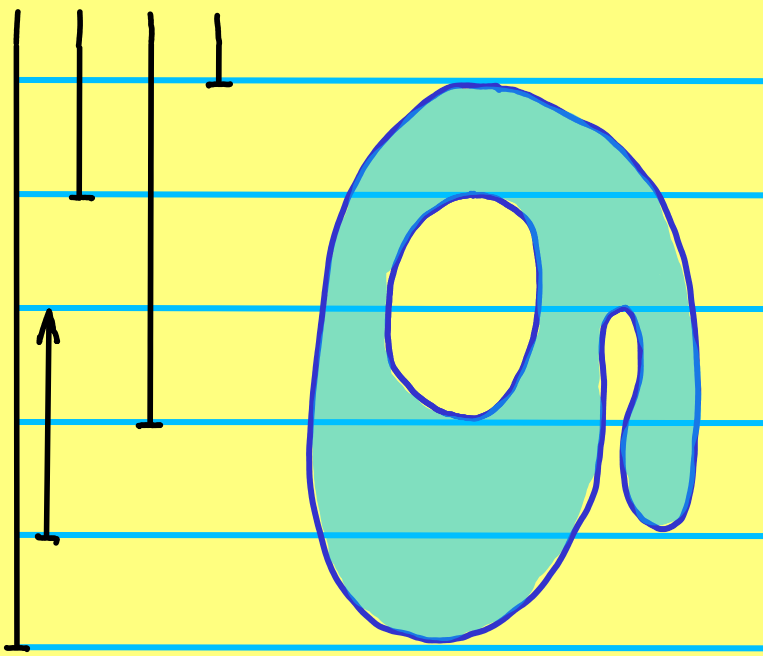
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	
	0 0 0	
	1 1 0	
1 1 0	2 2 0	
2 1 0	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



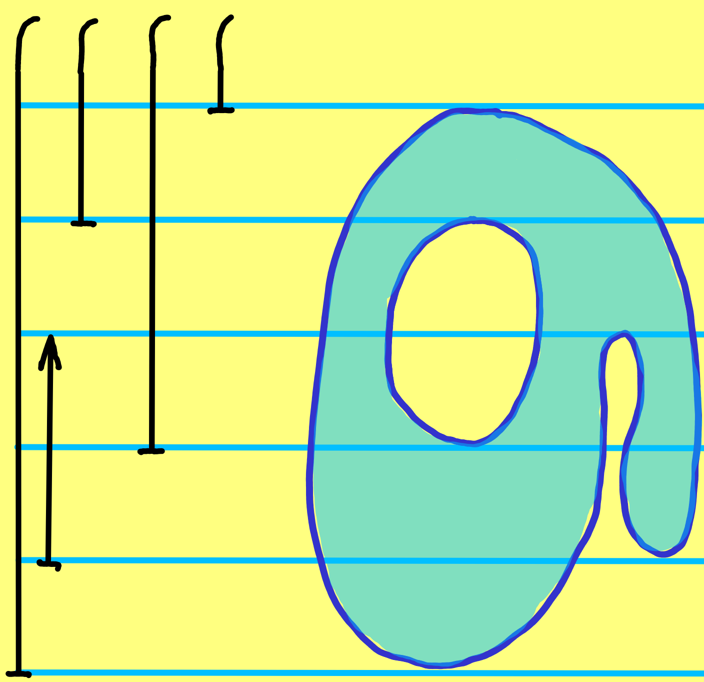
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	
	0 0 0	
1 2 0	1 1 0	
1 1 0	2 2 0	
2 1 0	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	
1 2 1	0 0 0	
1 2 0	1 1 0	
1 1 0	2 2 0	
2 1 0	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	superlevel set $f^{-1}[c, \infty)$
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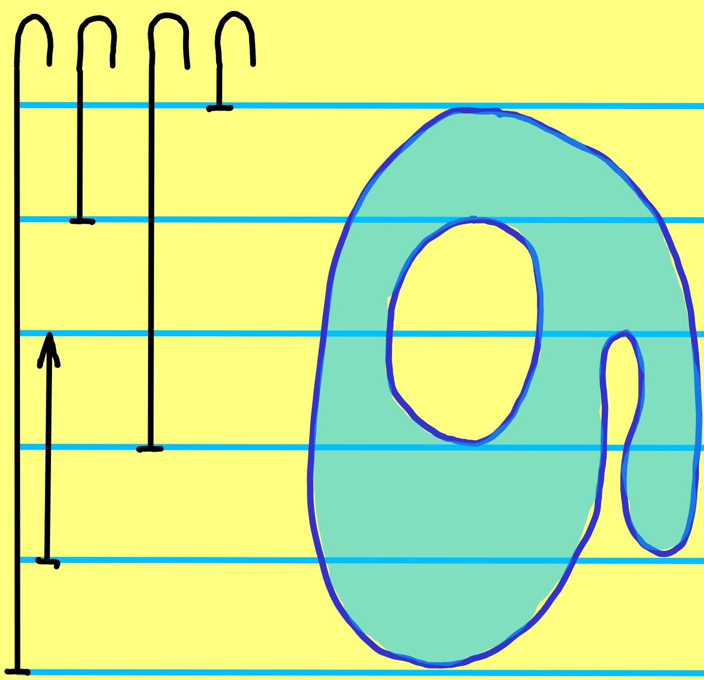
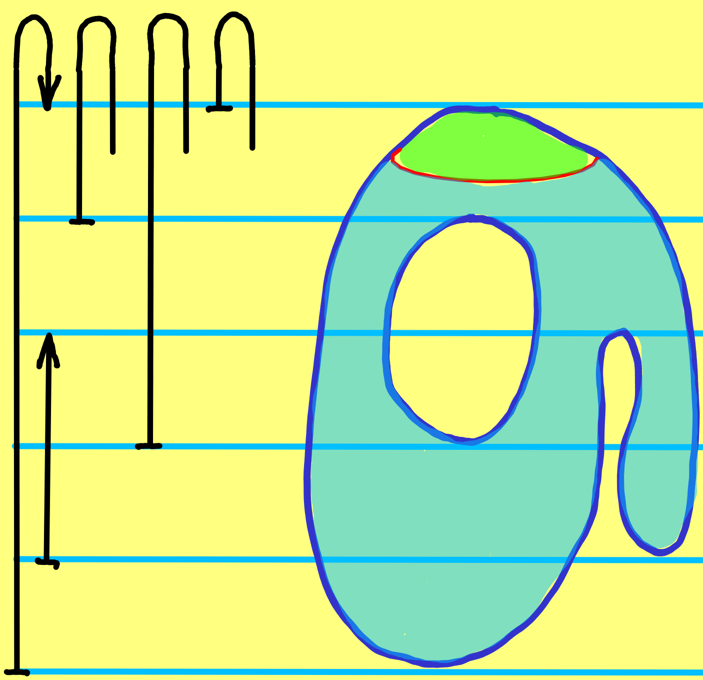
1 2 0	1 1 0	
1 1 0	2 2 0	
2 1 0	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



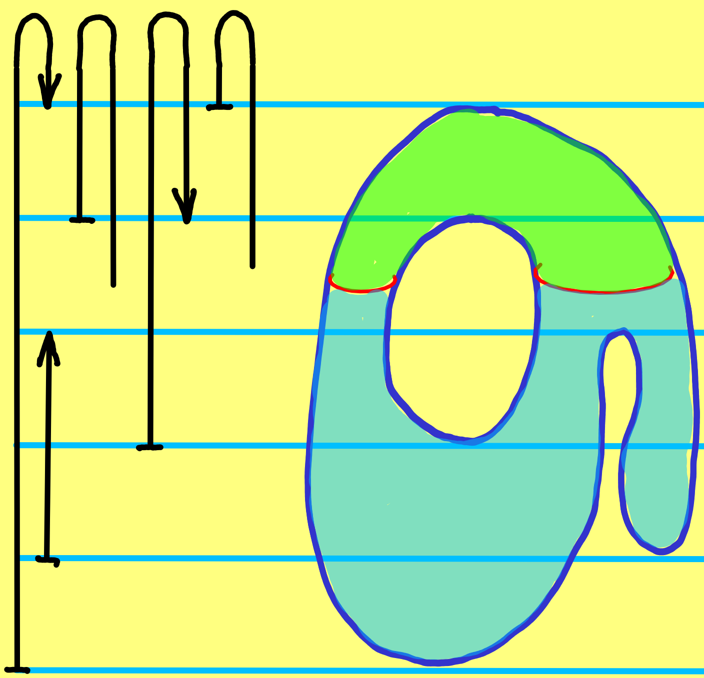
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	superlevel set $f^{-1}[c, \infty)$
1 2 1	0 0 0	1 2 1
1 2 0	1 1 0	
1 1 0	2 2 0	
2 1 0	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



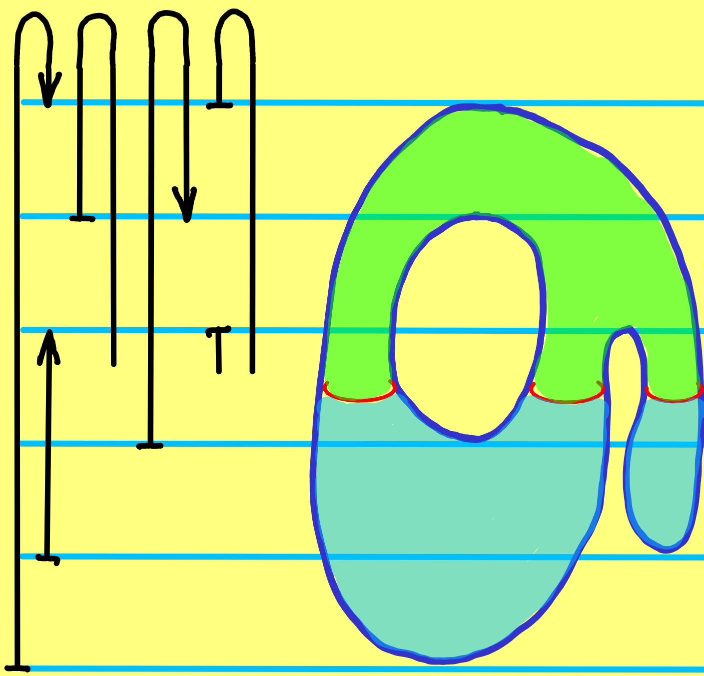
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	superlevel set $f^{-1}[c, \infty)$
1 2 1	0 0 0	1 2 1
1 2 0	1 1 0	0 2 1
1 1 0	2 2 0	
2 1 0	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



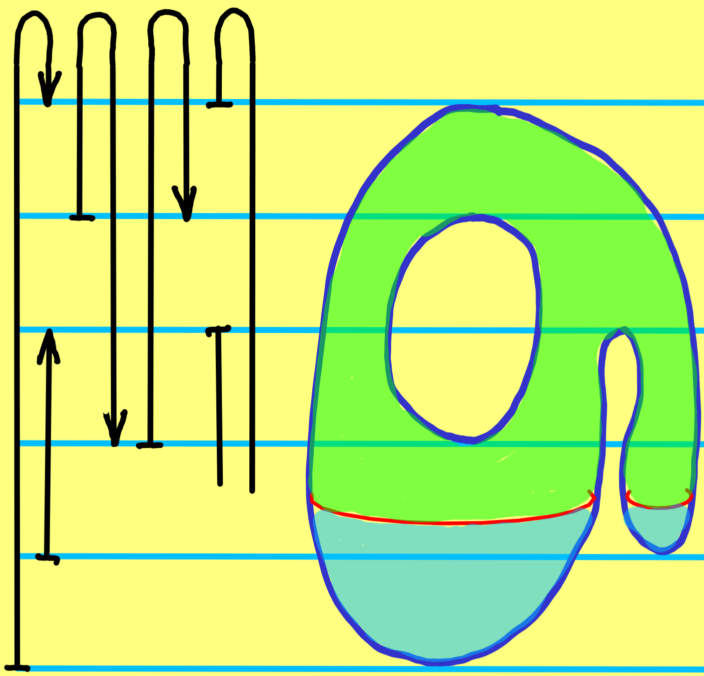
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	superlevel set $f^{-1}[c, \infty)$
1 2 1	0 0 0	1 2 1
1 2 0	1 1 0	0 2 1
1 1 0	2 2 0	0 1 1
2 1 0	3 3 0	
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



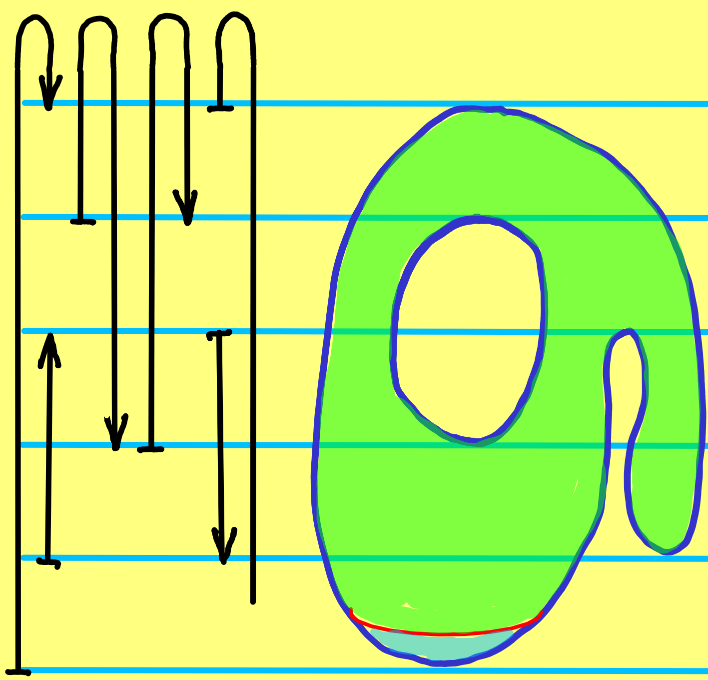
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	superlevel set $f^{-1}[c, \infty)$
1 2 1	0 0 0	1 2 1
1 2 0	1 1 0	0 2 1
1 1 0	2 2 0	0 1 1
2 1 0	3 3 0	0 1 2
2 0 0	2 2 0	
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



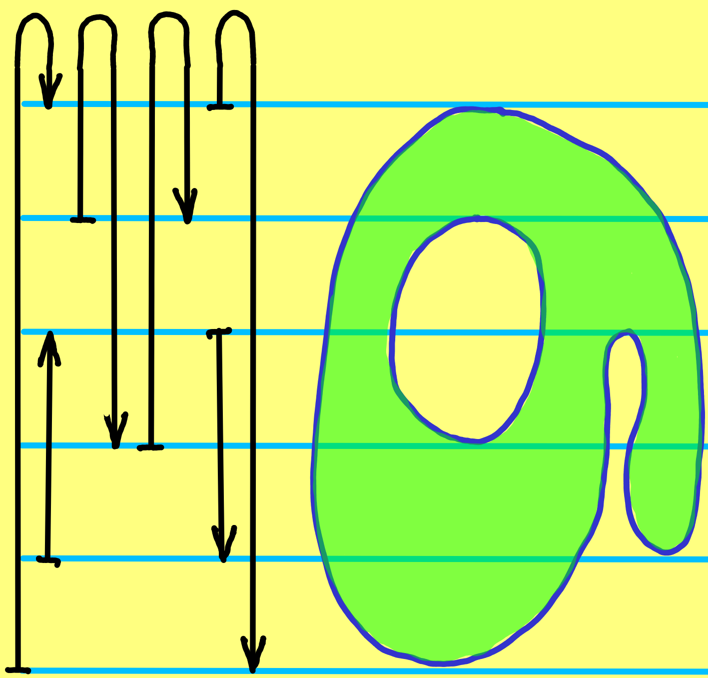
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	superlevel set $f^{-1}[c, \infty)$
1 2 1	0 0 0	1 2 1
1 2 0	1 1 0	0 2 1
1 1 0	2 2 0	0 1 1
2 1 0	3 3 0	0 1 2
2 0 0	2 2 0	0 0 2
1 0 0	1 1 0	
0 0 0	0 0 0	

Birth & Death



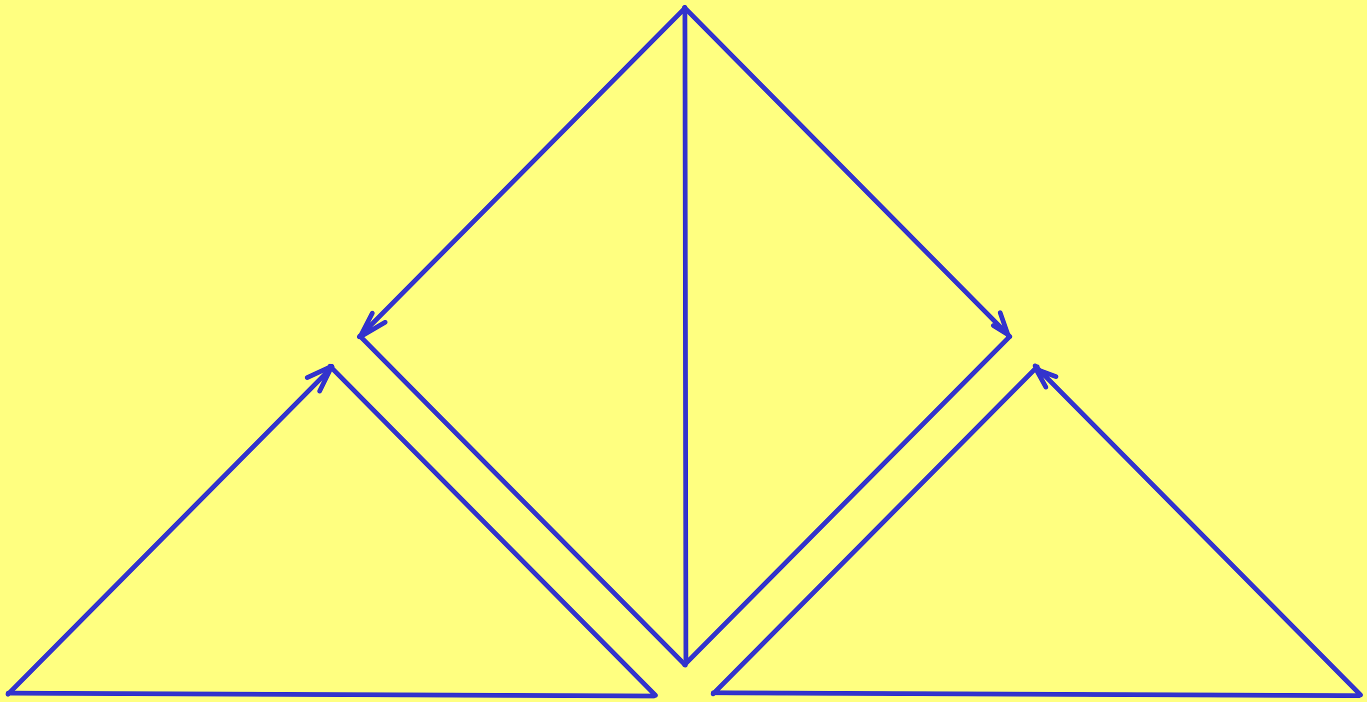
sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	superlevel set $f^{-1}[c, \infty)$
1 2 1	0 0 0	1 2 1
1 2 0	1 1 0	0 2 1
1 1 0	2 2 0	0 1 1
2 1 0	3 3 0	0 1 2
2 0 0	2 2 0	0 0 2
1 0 0	1 1 0	0 0 1
0 0 0	0 0 0	

Birth & Death

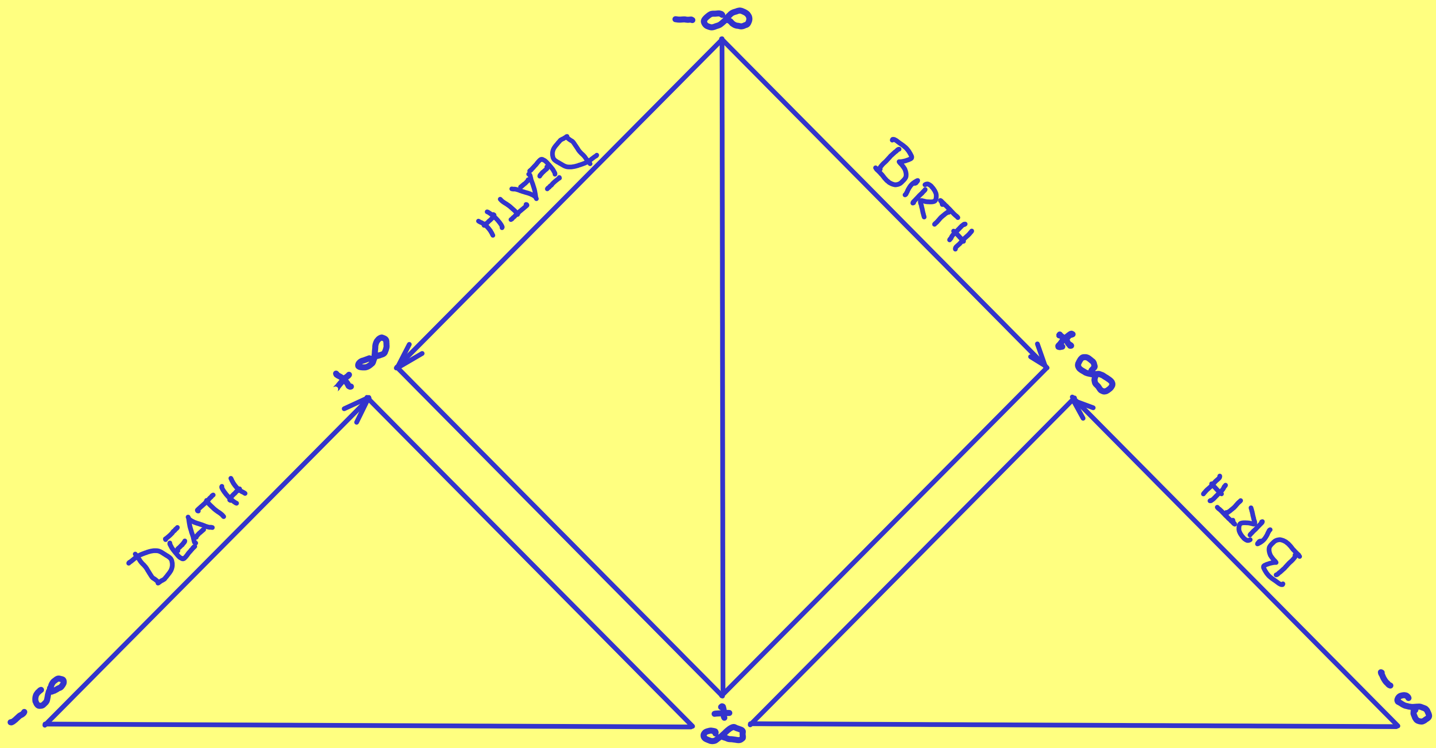


sublevel set $f^{-1}(-\infty, c]$	level set $f^{-1}(c)$	superlevel set $f^{-1}[c, \infty)$
1 2 1	0 0 0	1 2 1
1 2 0	1 1 0	0 2 1
1 1 0	2 2 0	0 1 1
2 1 0	3 3 0	0 1 2
2 0 0	2 2 0	0 0 2
1 0 0	1 1 0	0 0 1
0 0 0	0 0 0	0 0 0

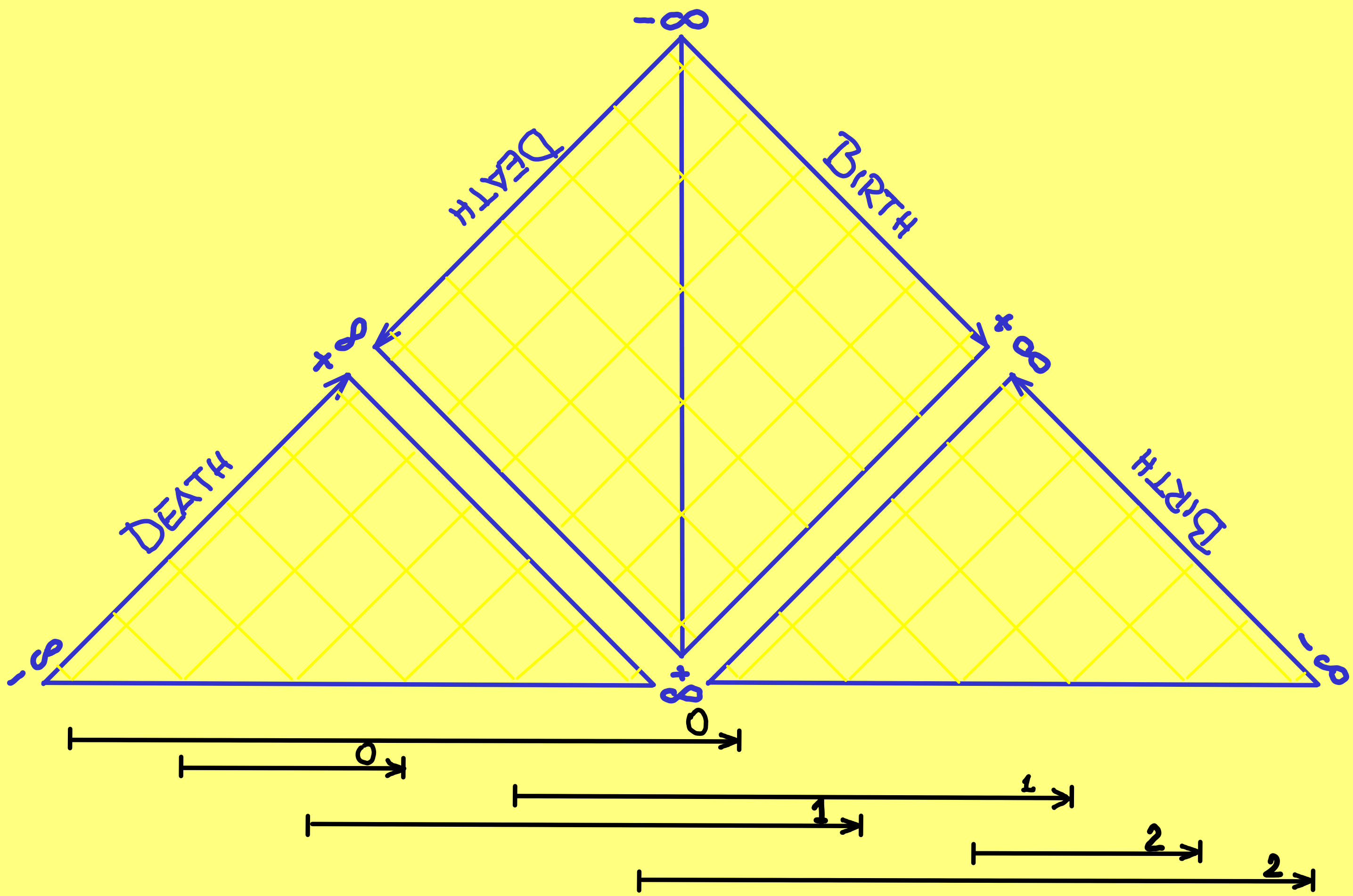
Persistence Diagram



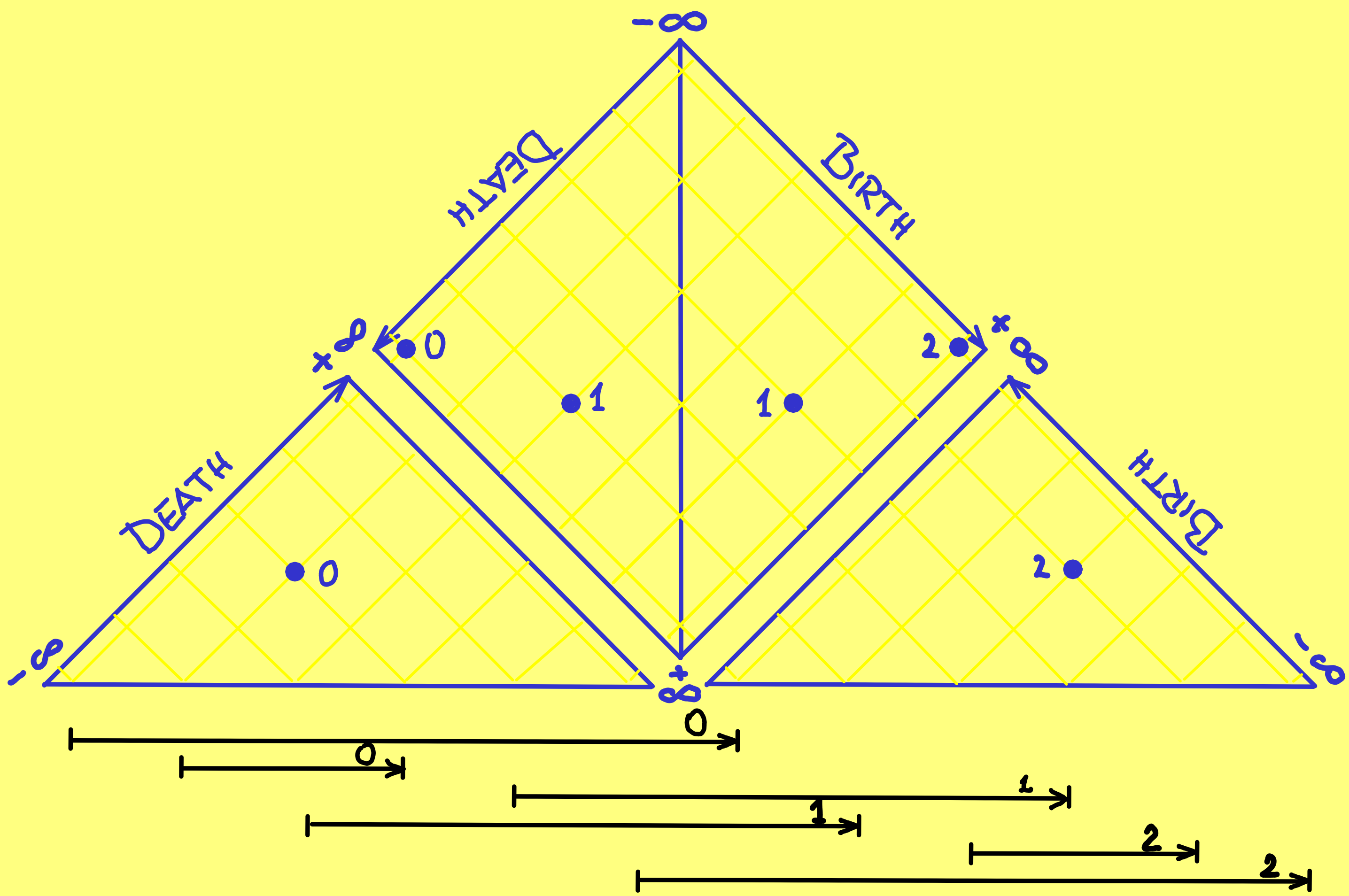
Persistence Diagram



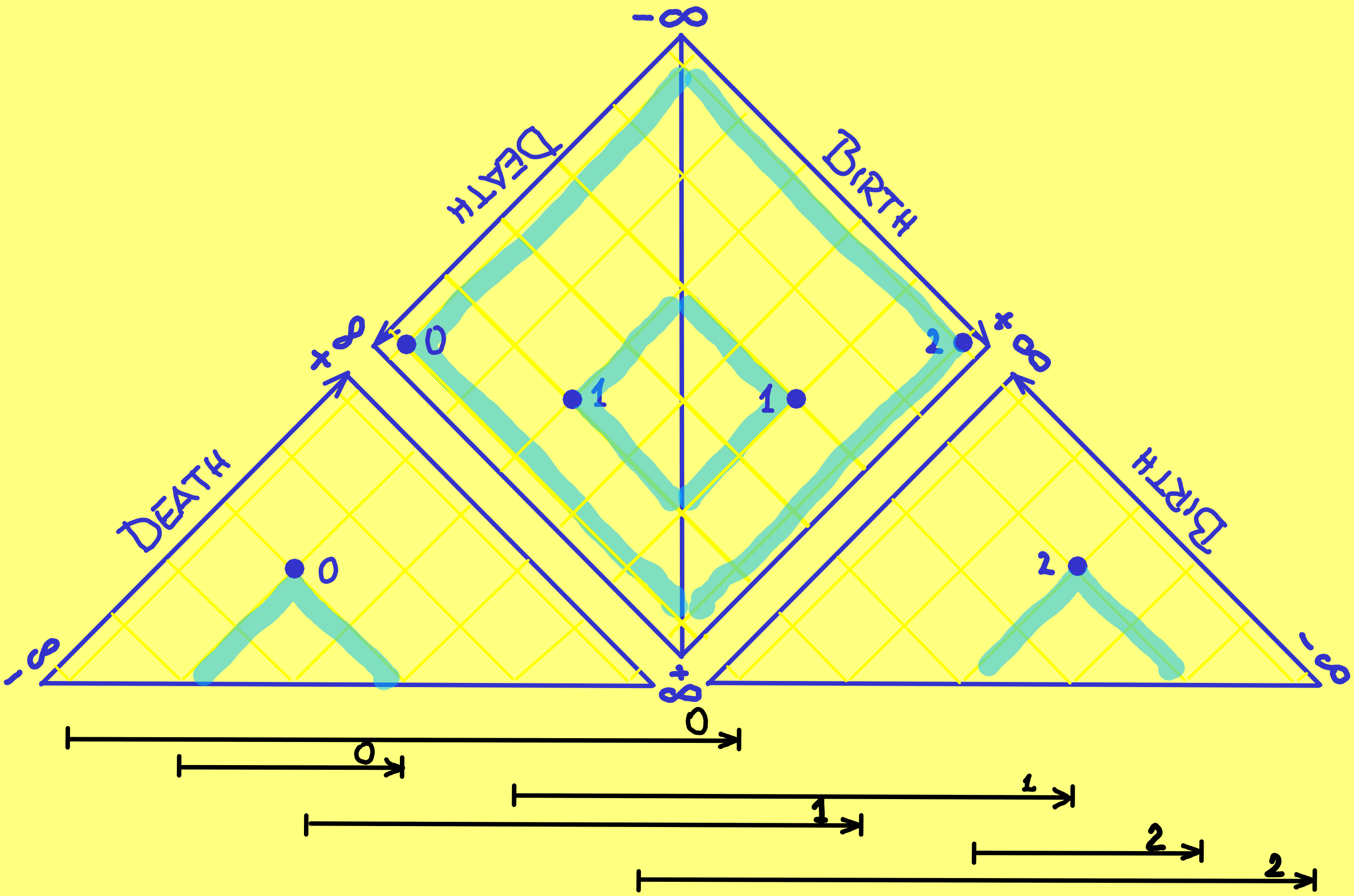
Persistence Diagram



Persistence Diagram

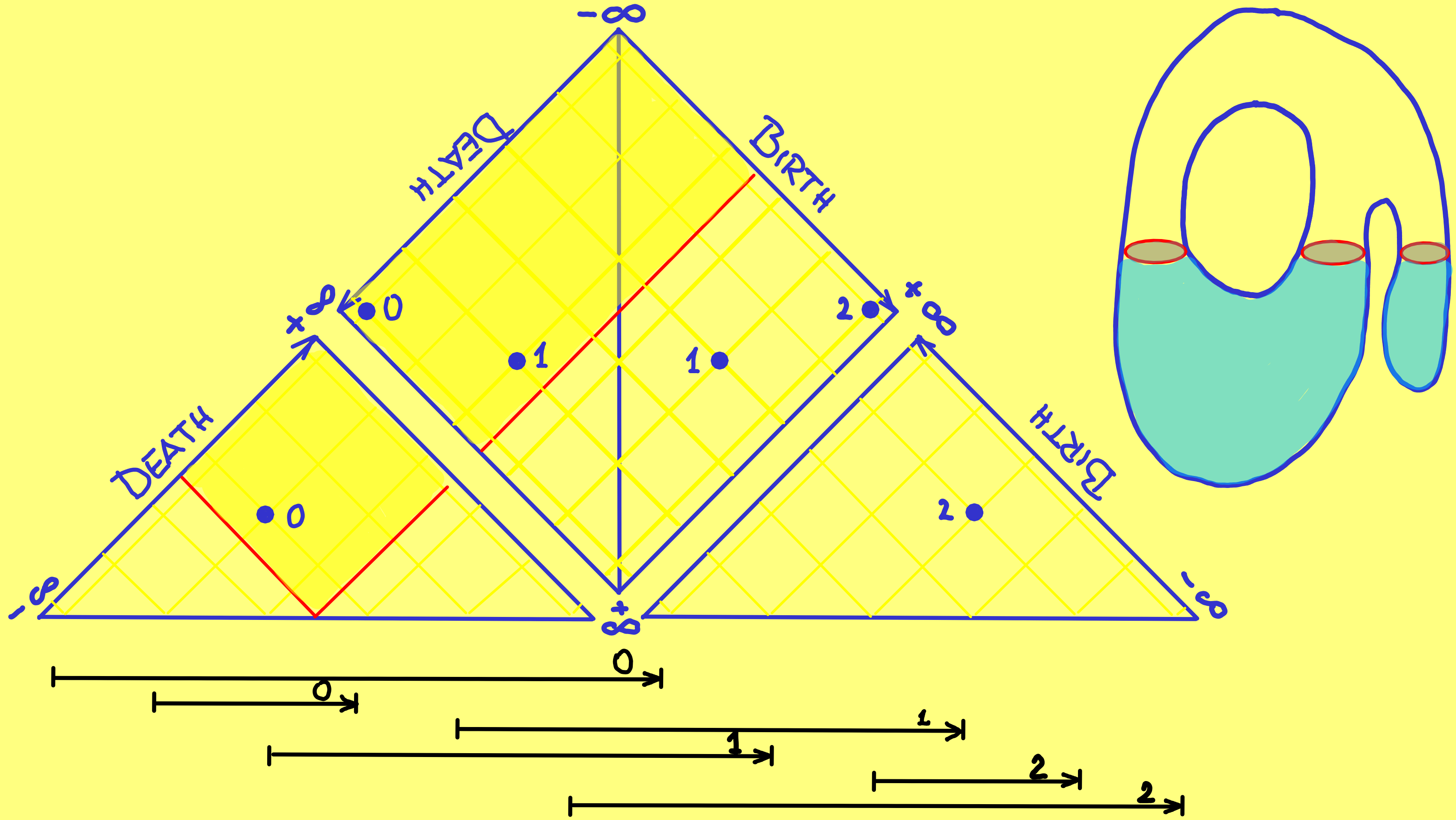


Persistence Diagram

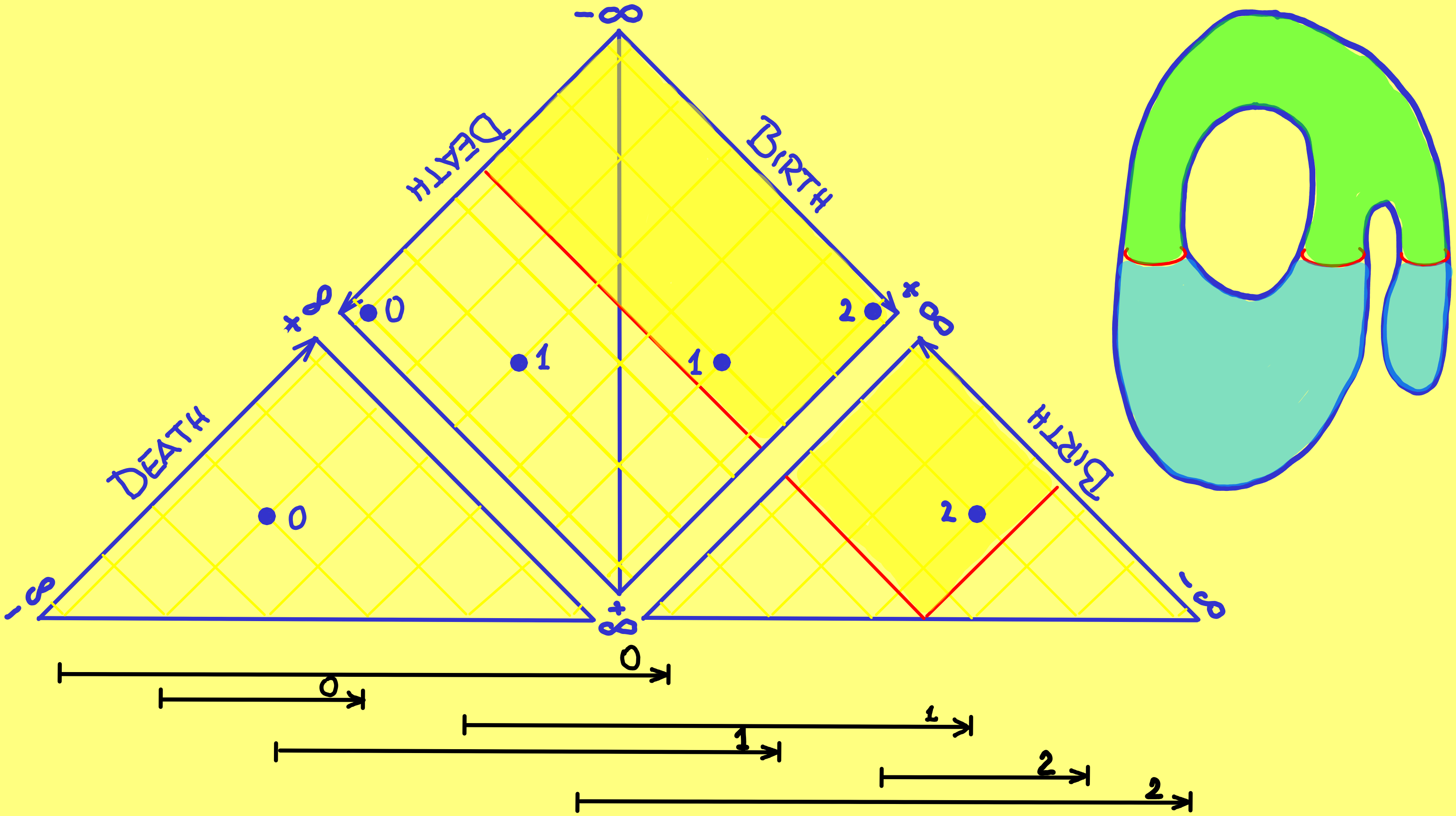


$A = (b, d), \quad \text{pers}(A) = |d - b|$

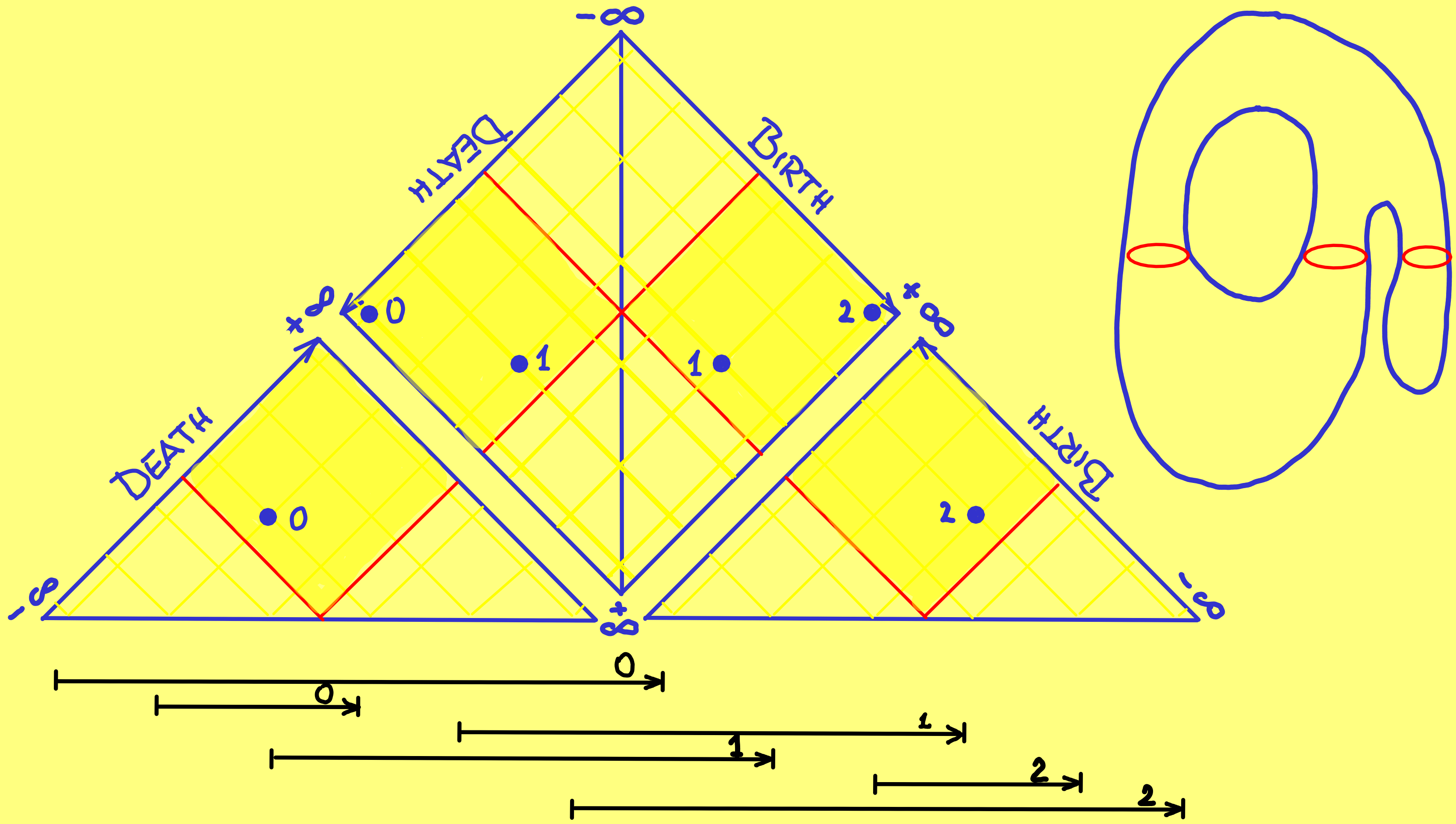
Persistence Diagram



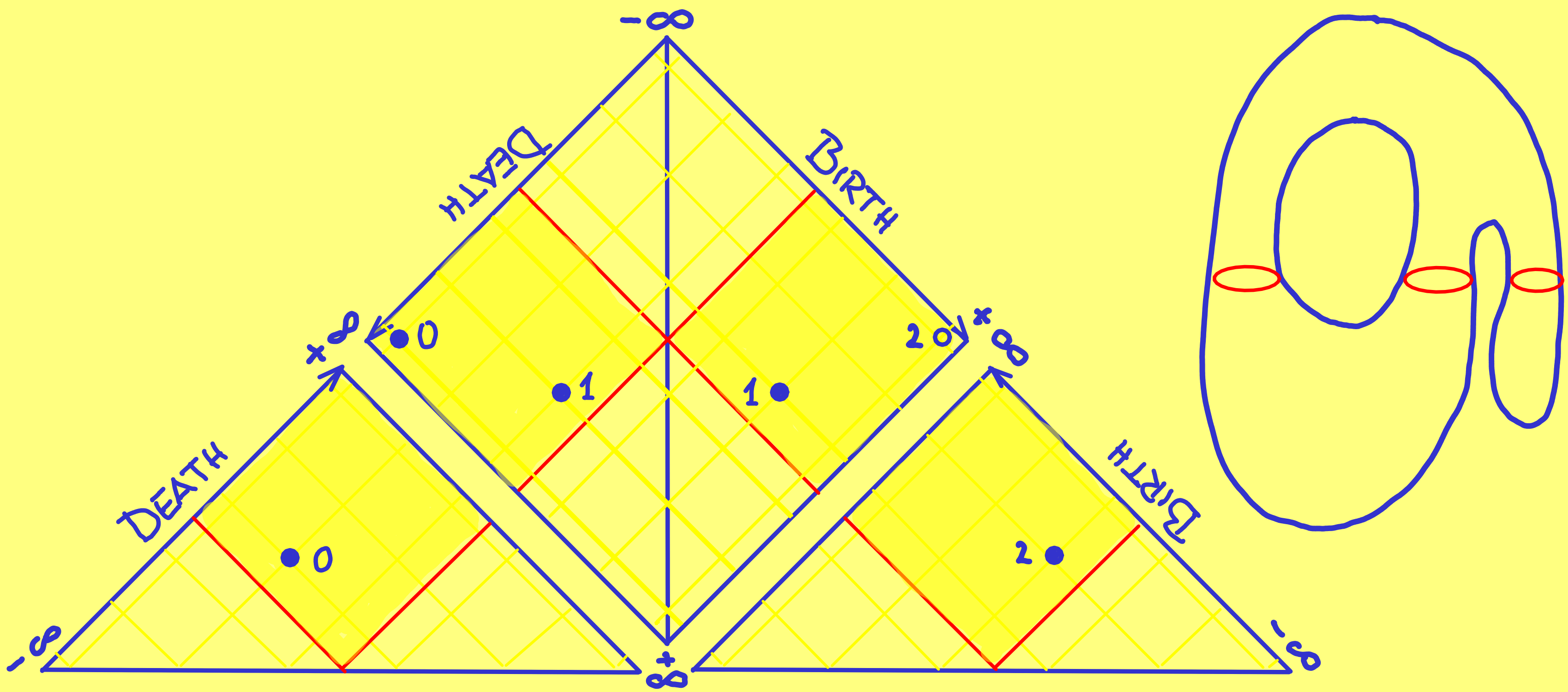
Persistence Diagram



Persistence Diagram



Moments

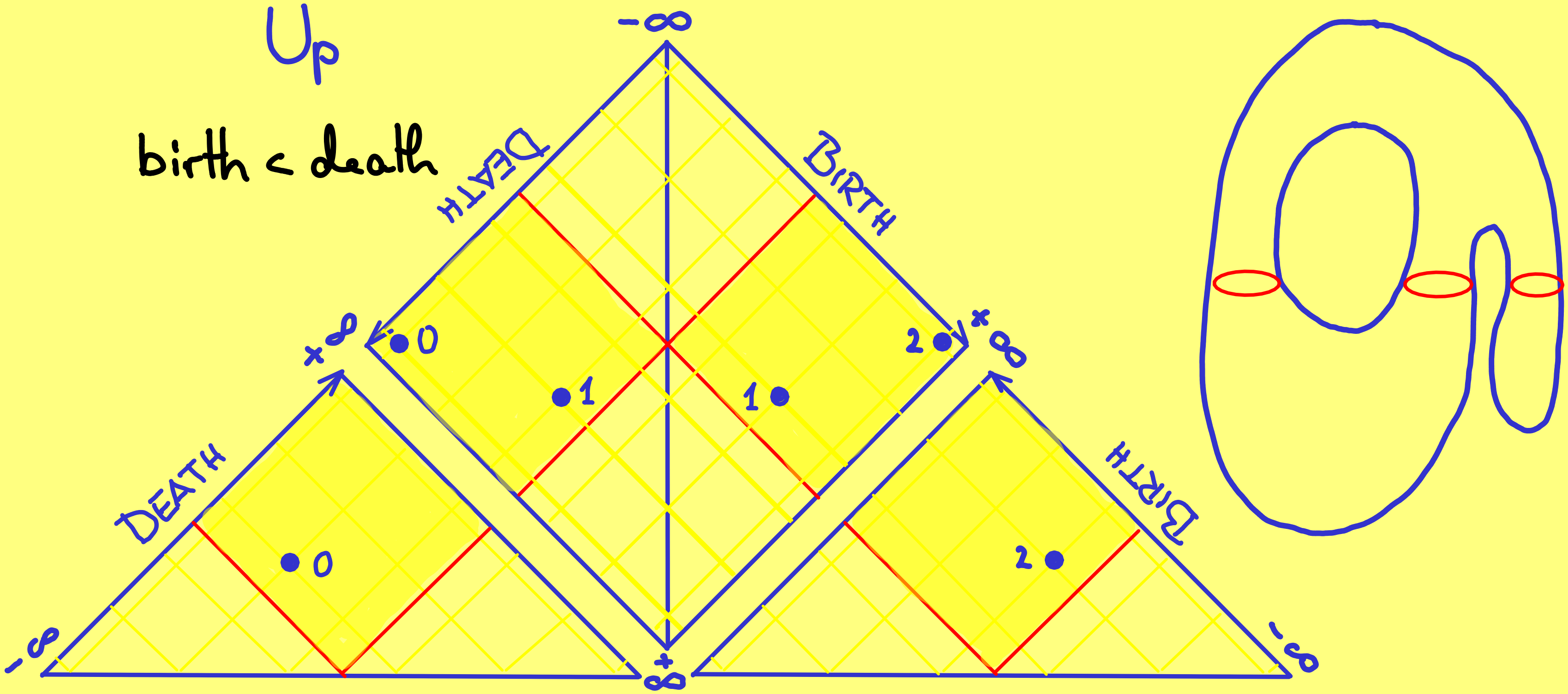


$-\infty$
DEATH
BIRTH
DEATH
BIRTH
$+\infty$
$+\infty$
$-\infty$
$-\infty$
$+\infty$
0
1
1
2
0
2

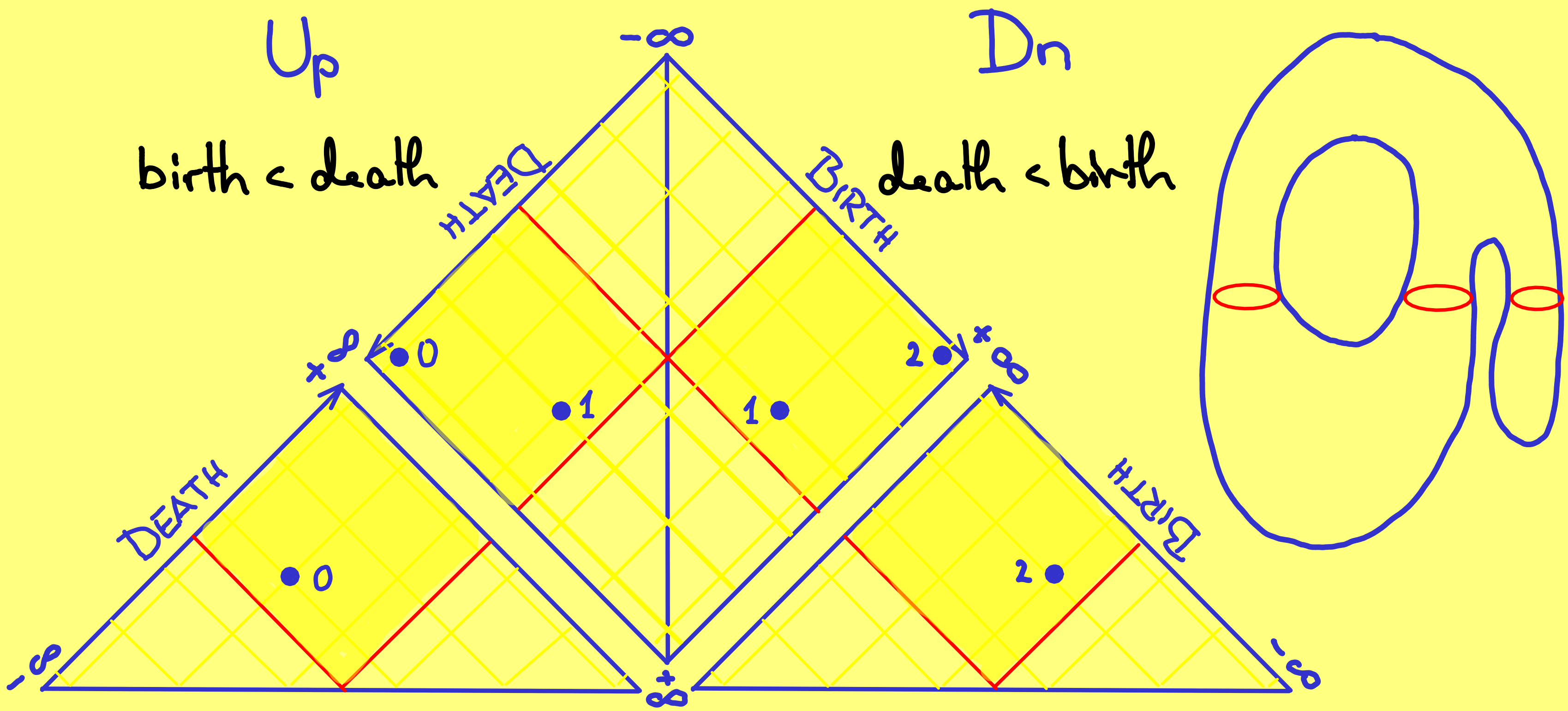
Moments

Up

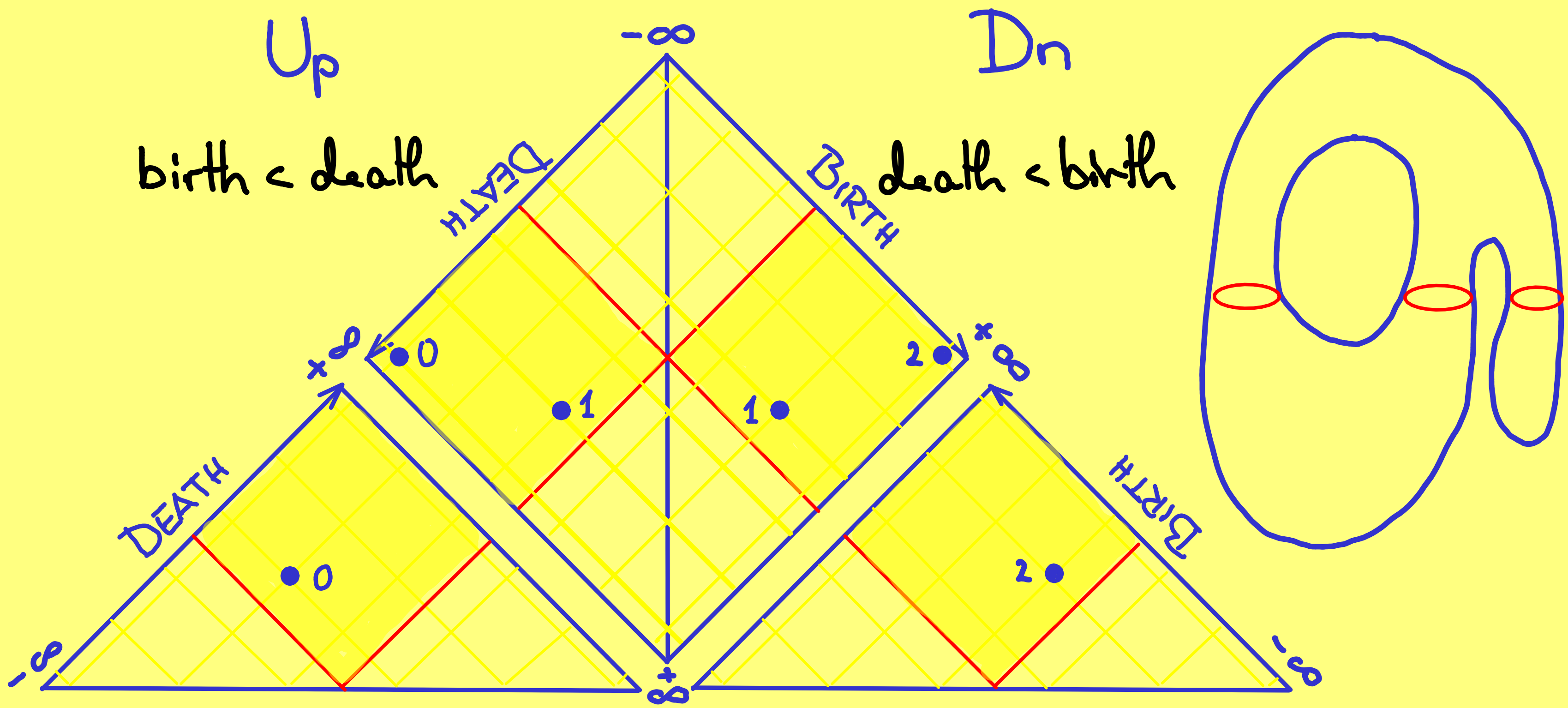
birth < death



Moments


Up
birth < death
Dn
death < birth
$-\infty$
$+\infty$
$-\infty$
$+\infty$
$+\infty$
$-\infty$
$+\infty$
DEATH
DEATH
BIRTH
BIRTH
0
1
0
2
1
2

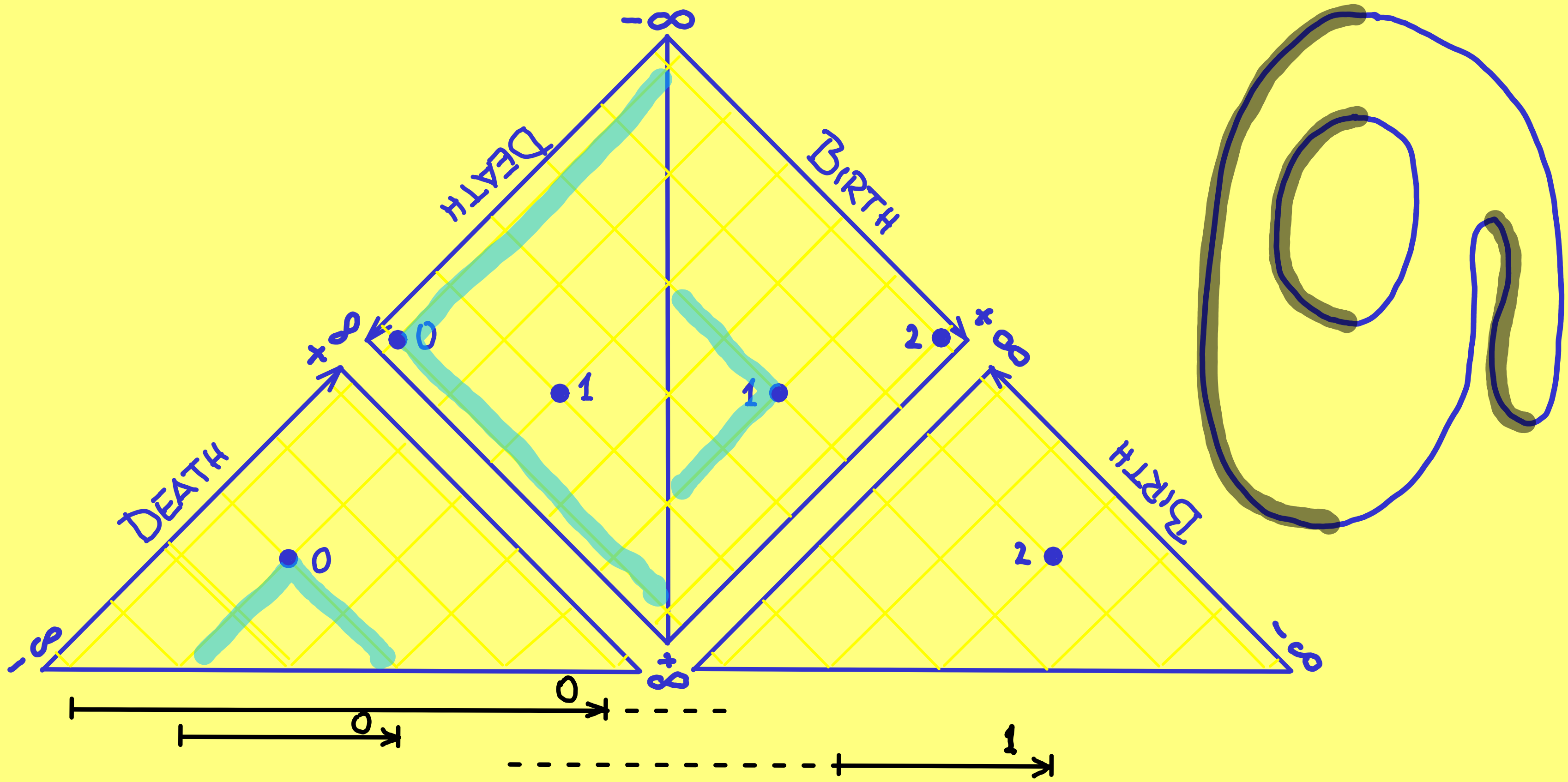
Moments



The p-th level set moment is

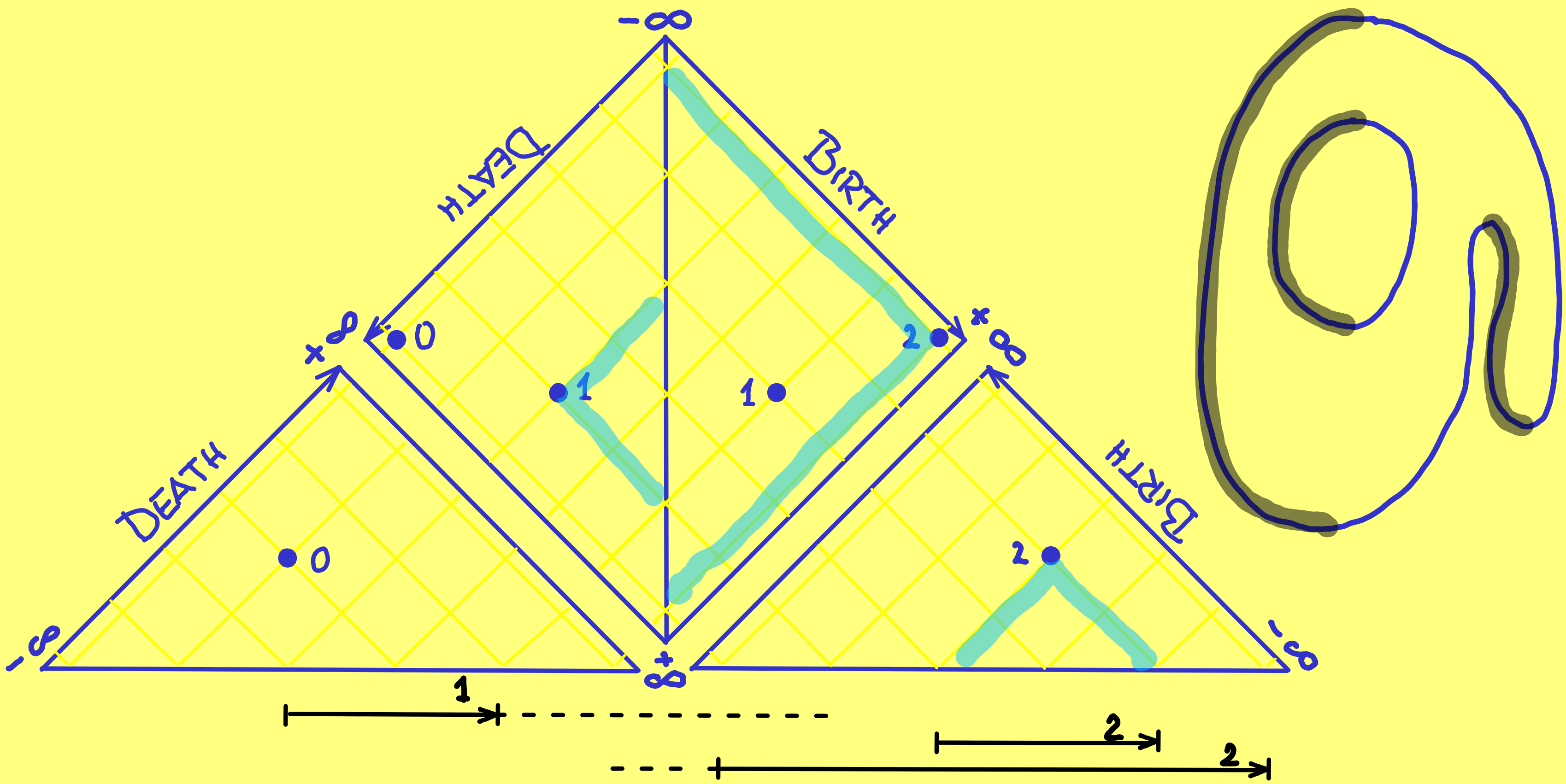
$$\sum_{A \in Up_p(f)} pers(A) + \sum_{A \in Dn_{p+1}(f)} pers(A).$$

MOMENTS



0-th level set moment

Moments



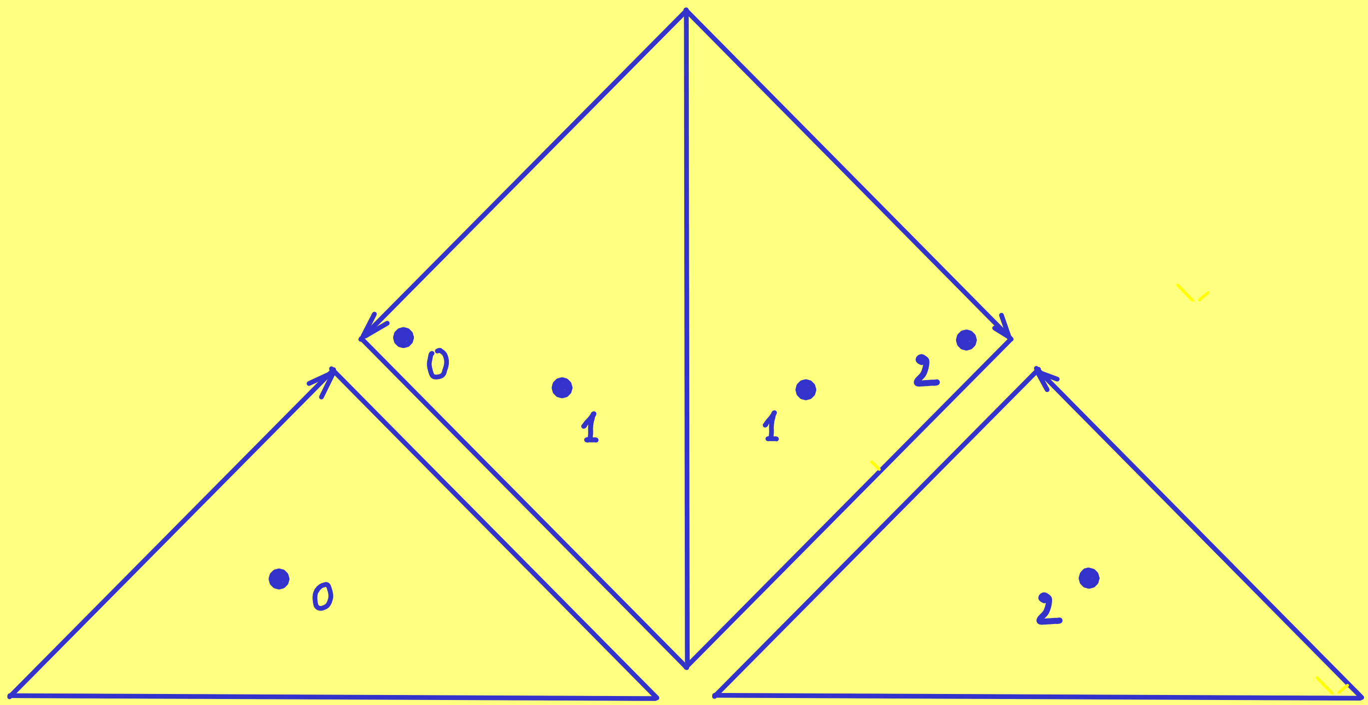
1-st level set moment

HISTORY

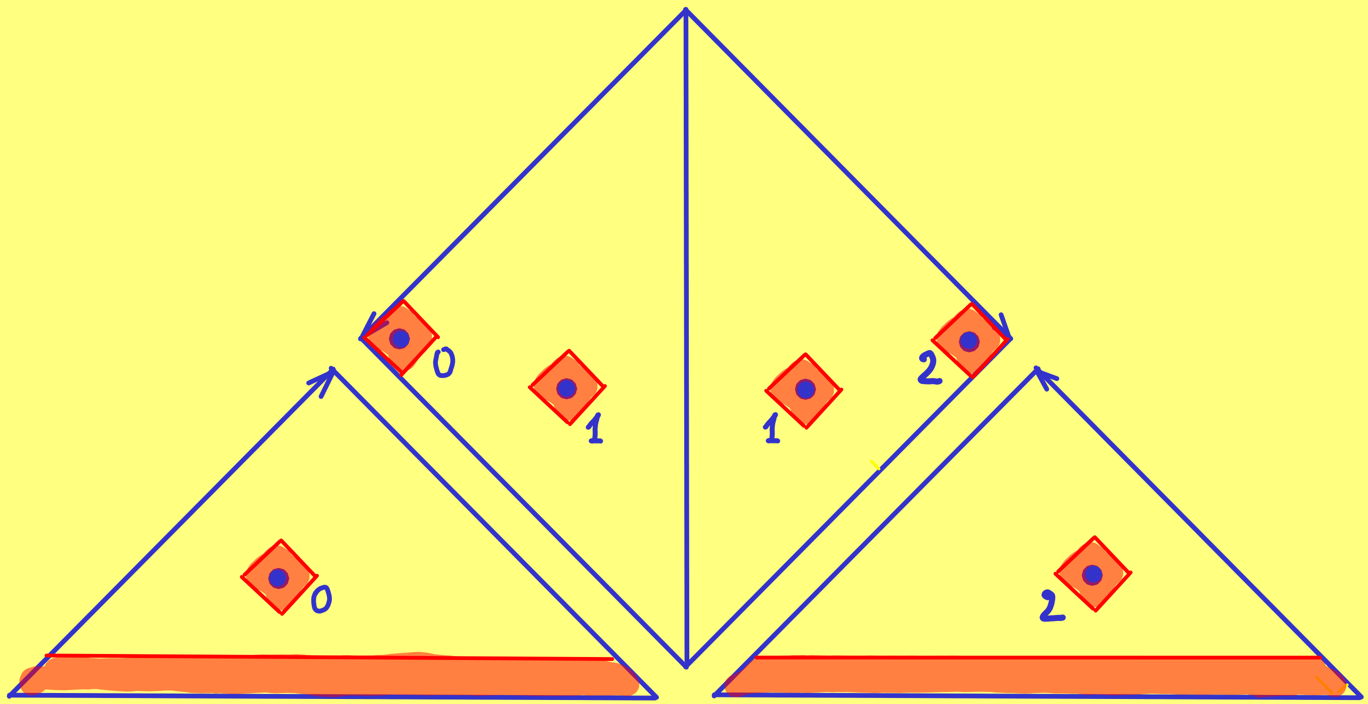
Persistent homology :	E, Letscher, Zomorodian	2002
Extended p.h. :	Cohen-Steiner, E, Harer	2007
Level sets :	Bendich, E, Morozov, Patel	2009

Shape, Homology, Persistence, and Stability

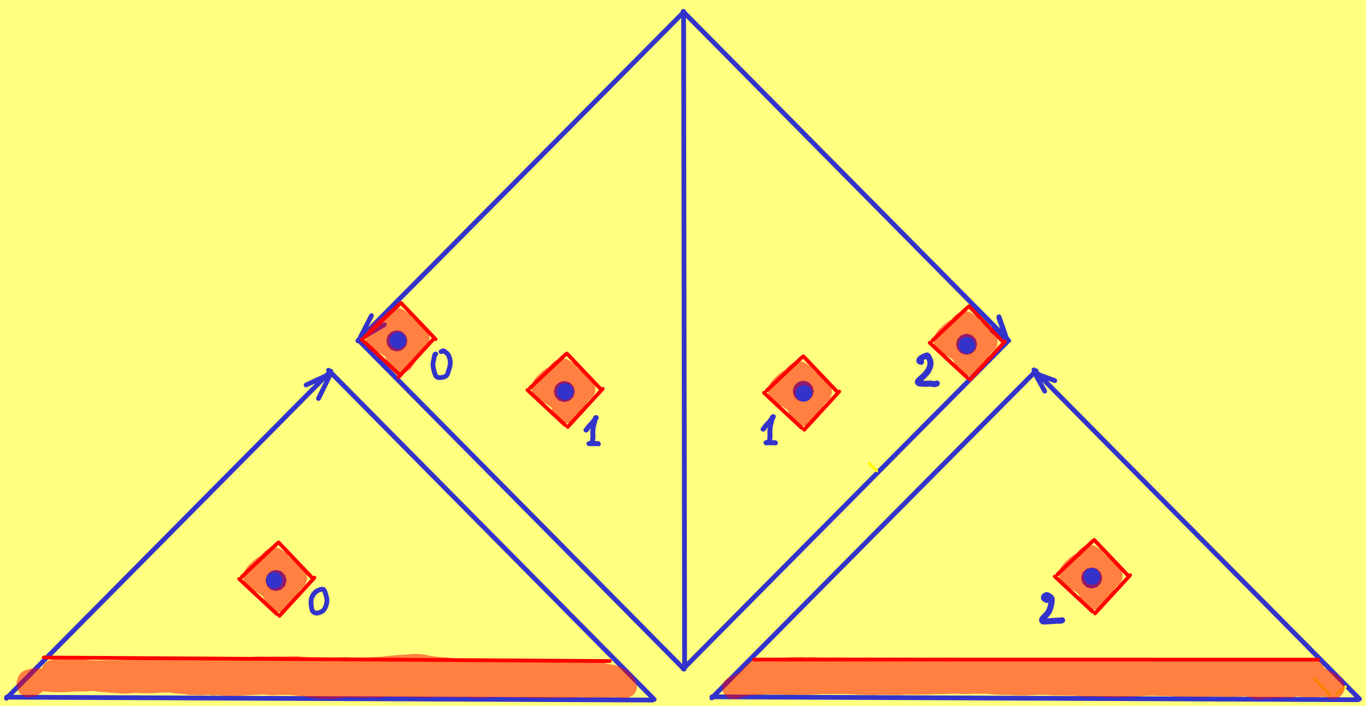
Bottleneck Stability



Bottleneck Stability

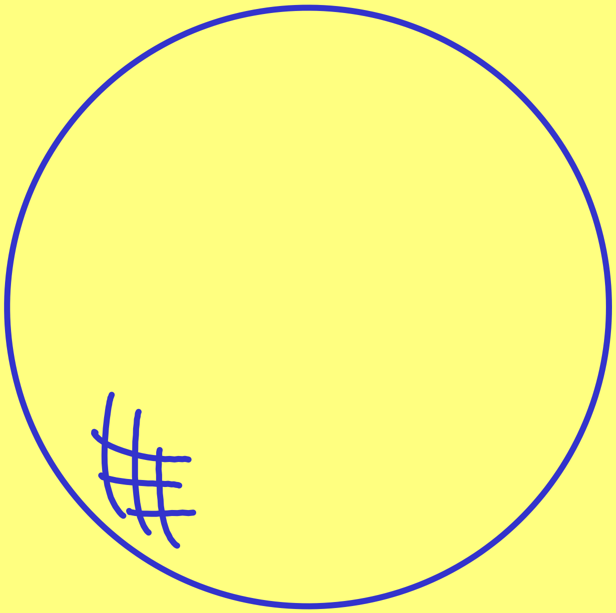


Bottleneck Stability



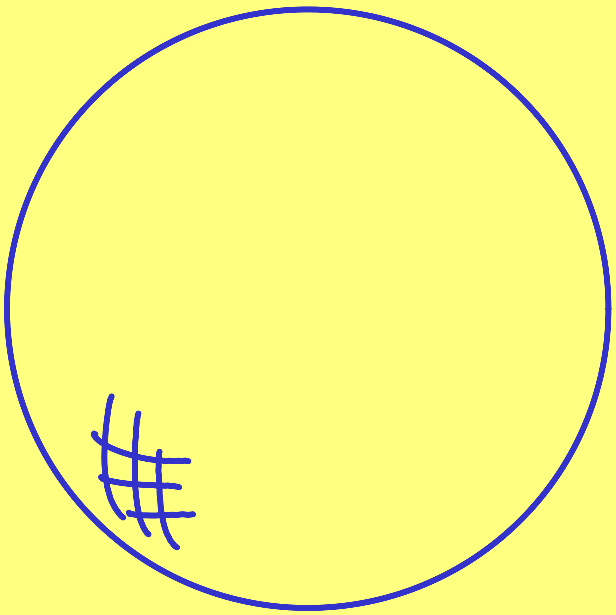
THM. $W_\infty(\mathrm{Dgm}(f), \mathrm{Dgm}(g)) \leq \|f-g\|_\infty.$

Volume of Unit Ball

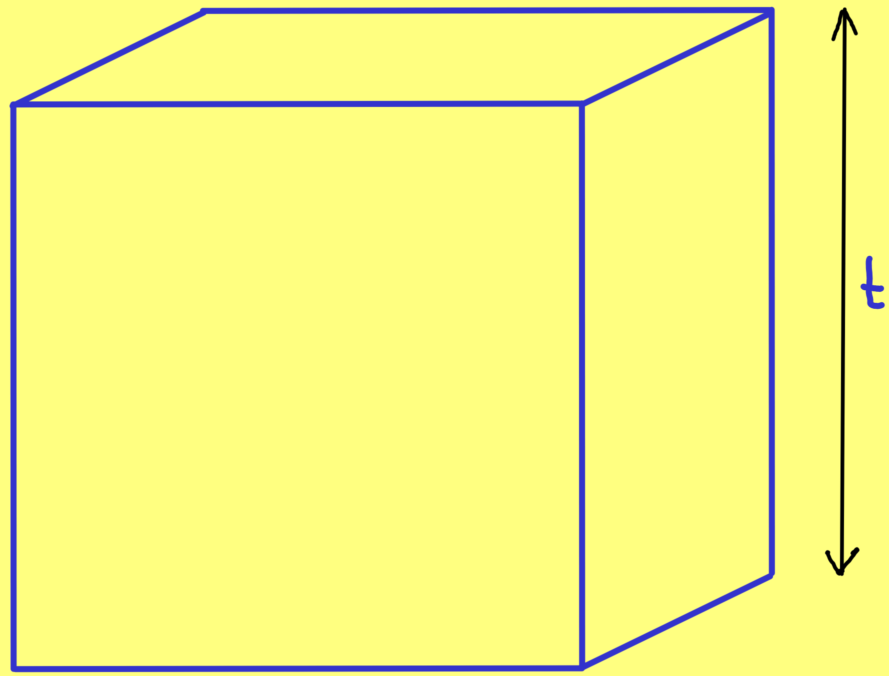


$\mathbb{B}^3 : \ \|x\| \leq 1.$

Volume of Unit Ball

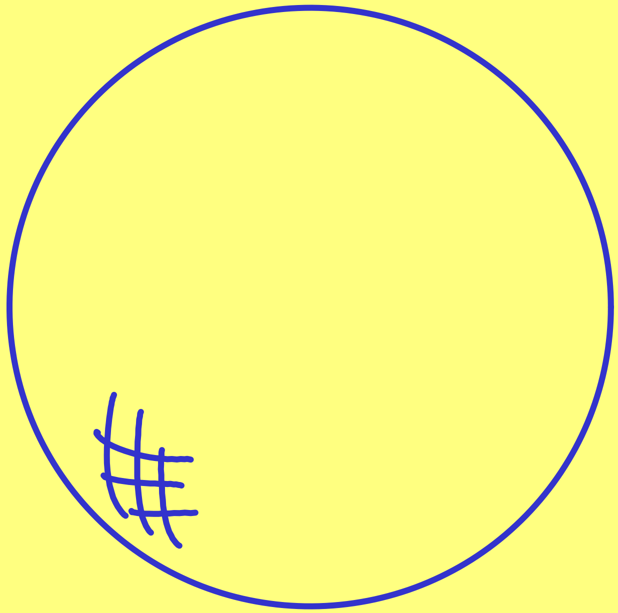


$\mathbb{B}^3 : \ \|x\| \leq 1.$

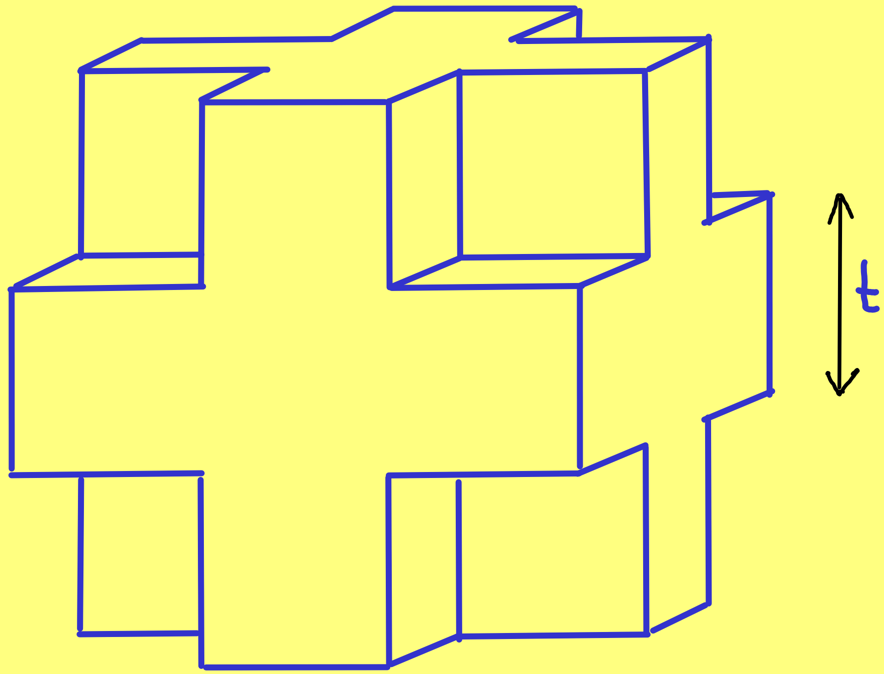


$\mathbb{B}^3_t$ is resolution-t approximation

Volume of Unit Ball

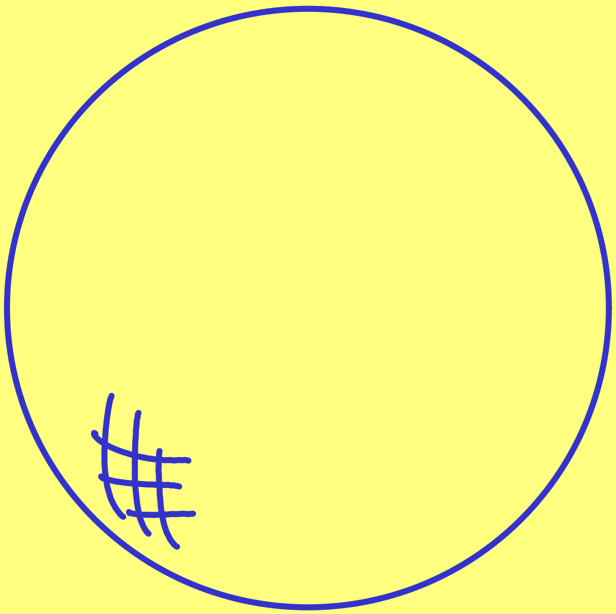


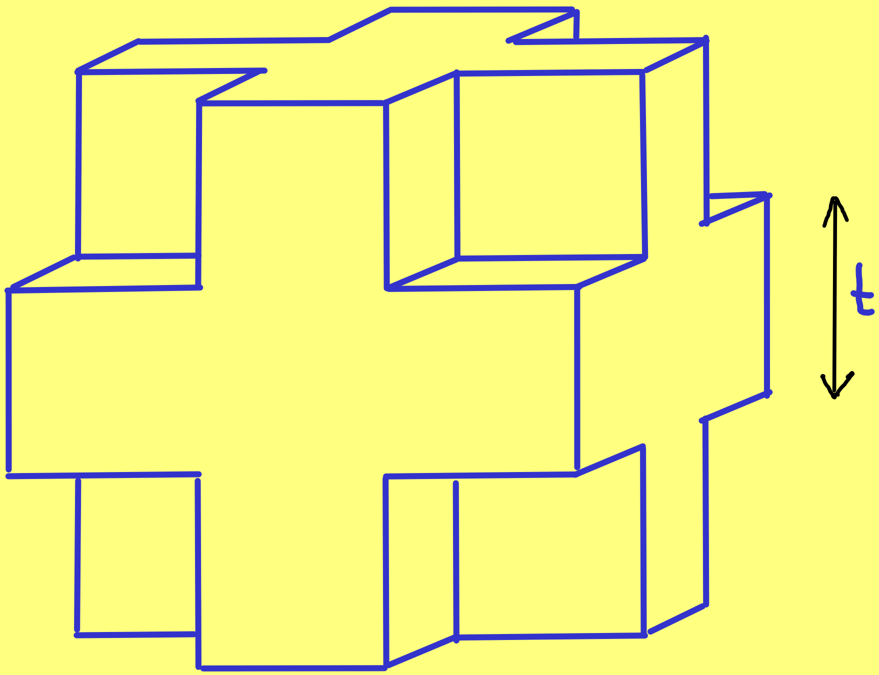
$\mathbb{B}^3 : \|x\| \leq 1.$



$\mathbb{B}^3_t$ is resolution-t approximation

Volume of Unit Ball



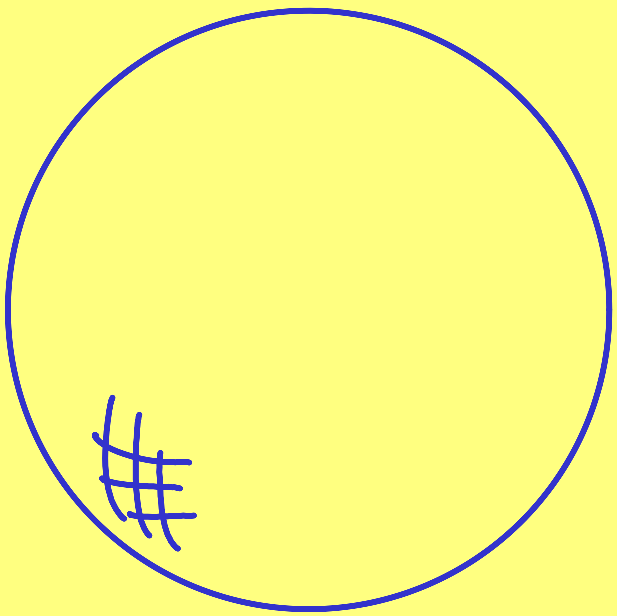


$\mathbb{B}^3 : \|x\| \leq 1.$

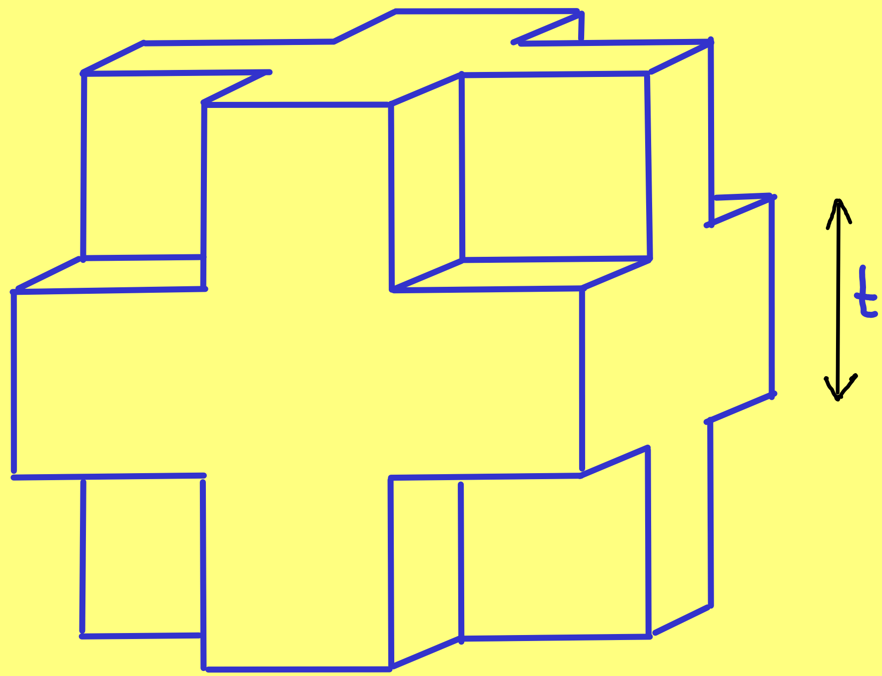
$\mathbb{B}^3_t$ is resolution-t approximation

$$\lim_{t\to 0} \mathrm{Vol}(\mathbb{B}^3_t) = \lim_{t\to 0} t^3 \#(\mathbb{B}^3 \cap t\mathbb{Z}^3) = \mathrm{Vol}(\mathbb{B}^3) = \frac{4}{3}\pi$$

Area of Unit Ball

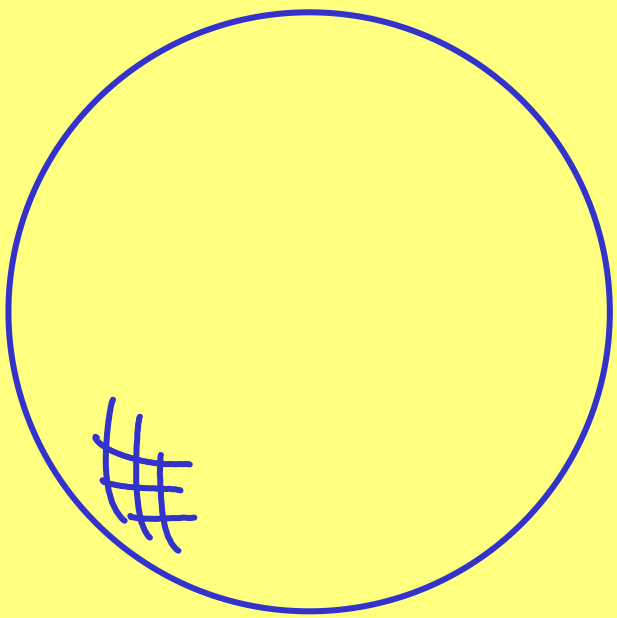


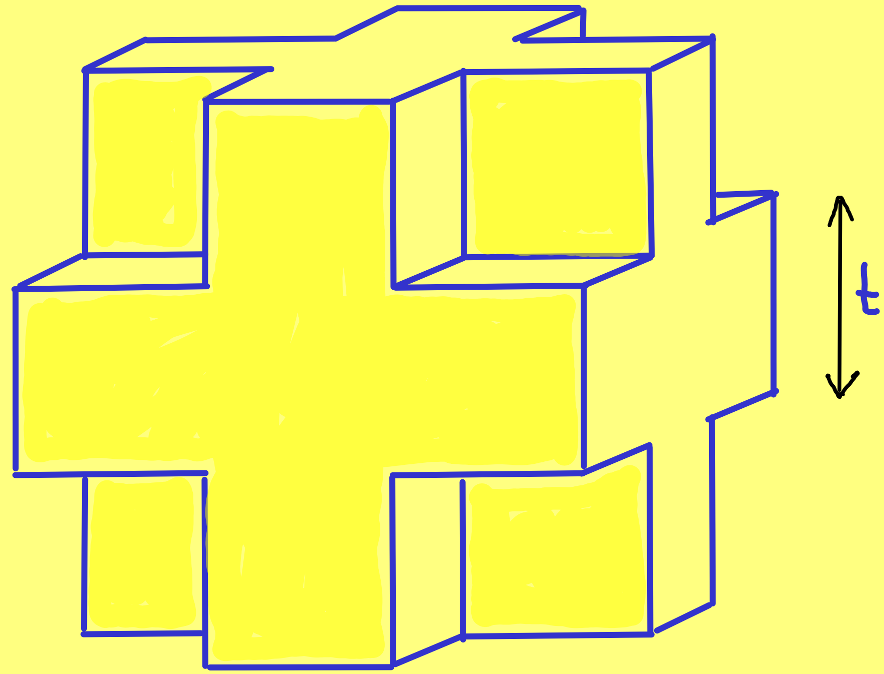
$\mathbb{B}^3 : \|x\| \le 1.$



$\mathbb{B}^3_t$ is resolution-t approximation

AREA OF UNIT BALL



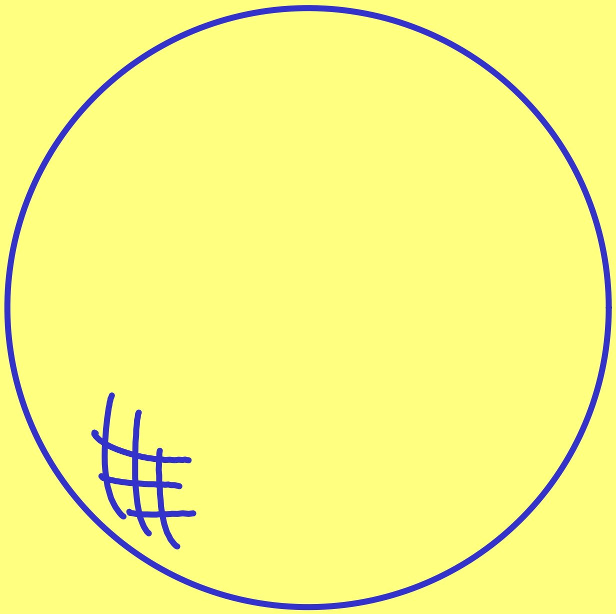


$\mathbb{B}^3 : \|x\| \leq 1.$

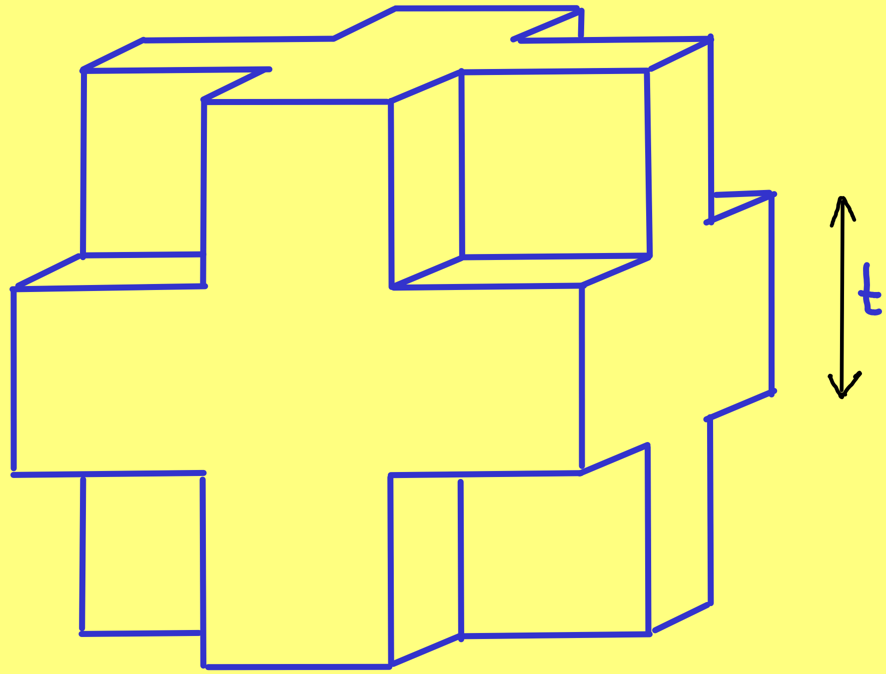
$\mathbb{B}^3_t$ is resolution-t approximation

$$\lim_{t\to 0} \text{Area}(\mathbb{B}^3_t) = \lim_{t\to 0} 6t^2 \#(\mathbb{B}^2 \cap t\mathbb{Z}^2) = 6\,\text{Area}(\mathbb{B}^2) = 6\pi$$

Mean Curvature of Unit Ball

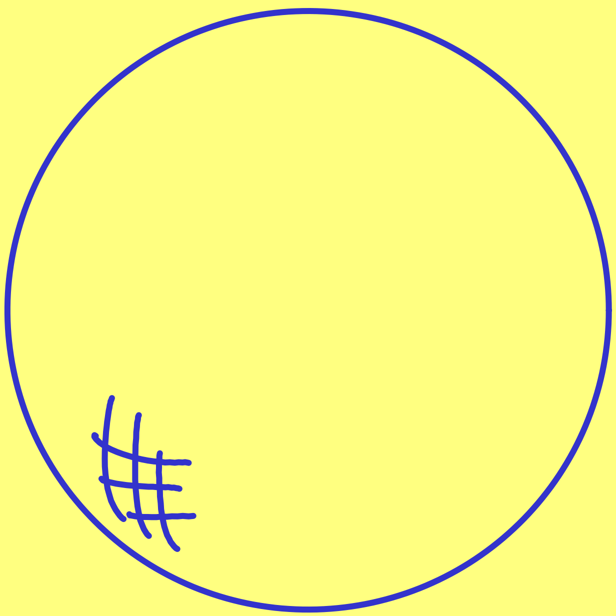


$\mathbb{B}^3 : \|x\| \leq 1.$

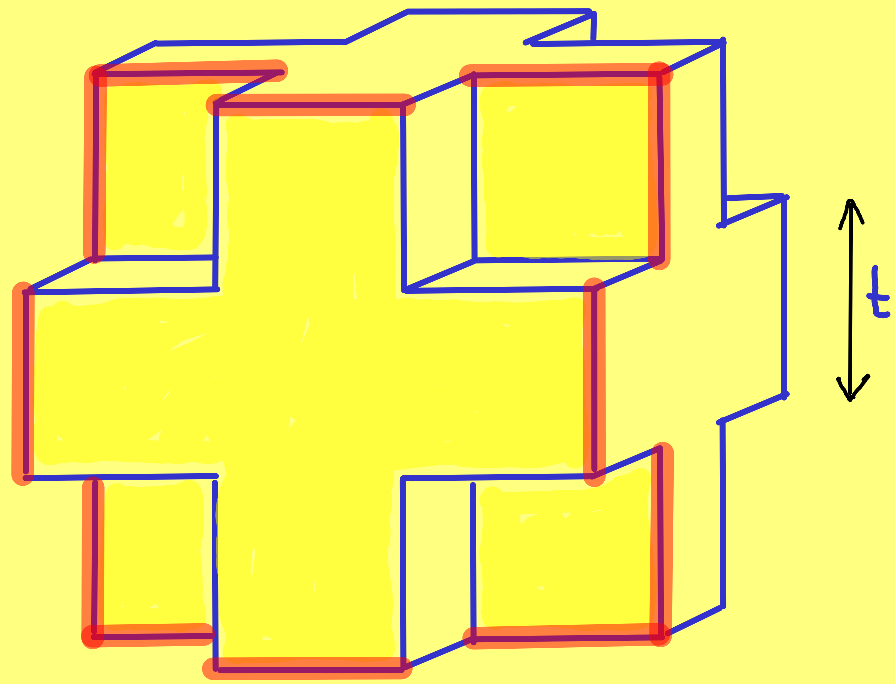


$\mathbb{B}^3_t$ is resolution-t approximation

Mean Curvature of Unit Ball



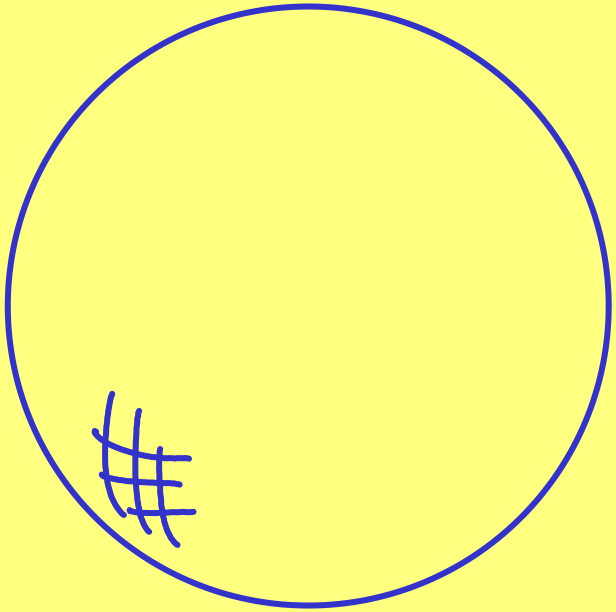
$\mathbb{B}^3 : \|x\| \leq 1.$



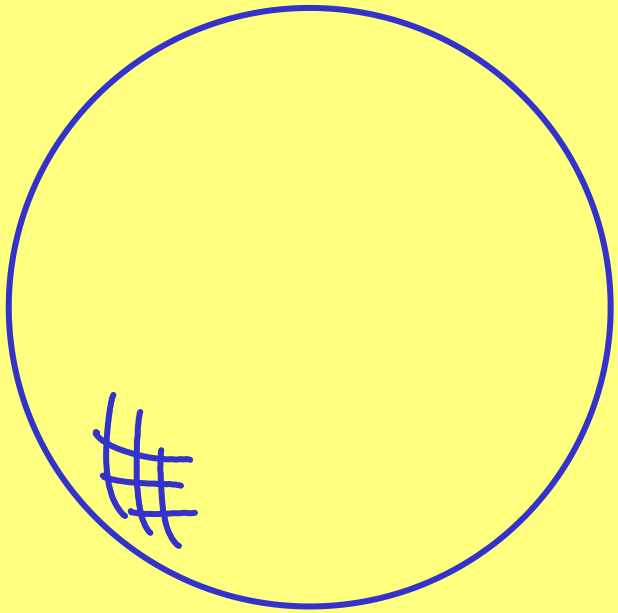
$\mathbb{B}^3_t$ is resolution-t approximation

$$\lim_{t \to 0} \mathrm{Mean}(\mathbb{B}^3_t) = \lim_{t \to 0} 3\pi t \,\#(\mathbb{B}^1 \cap t\mathbb{Z}^1) = 3\pi \,\mathrm{Length}(\mathbb{B}^1) = 6\pi$$

Convergence



Convergence

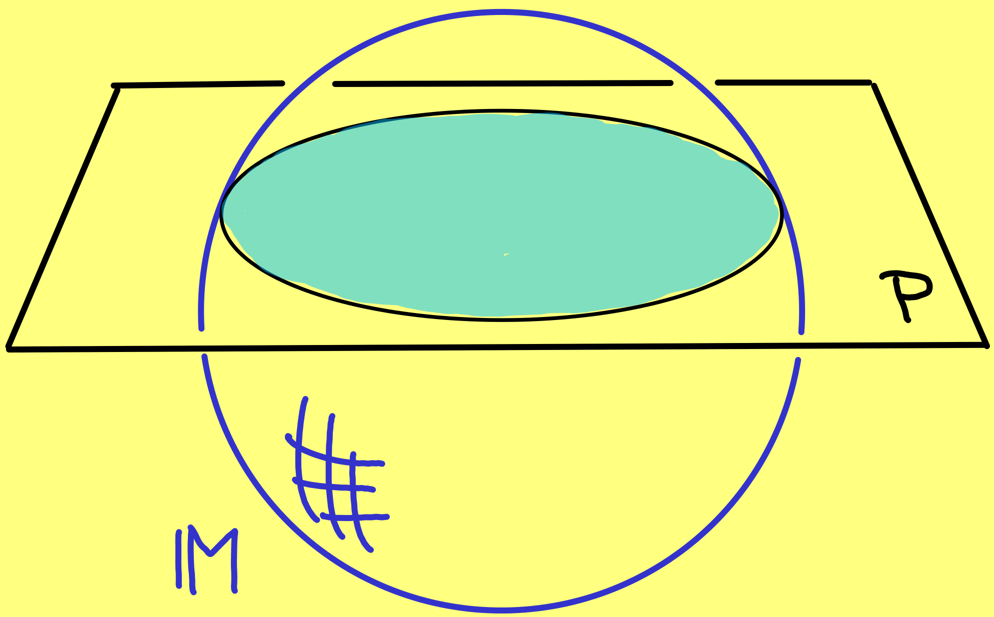


solid body $\mathbb{M} \subseteq \mathbb{R}^3$

that is : compact and

smoothly embed. $\partial \mathbb{M}$

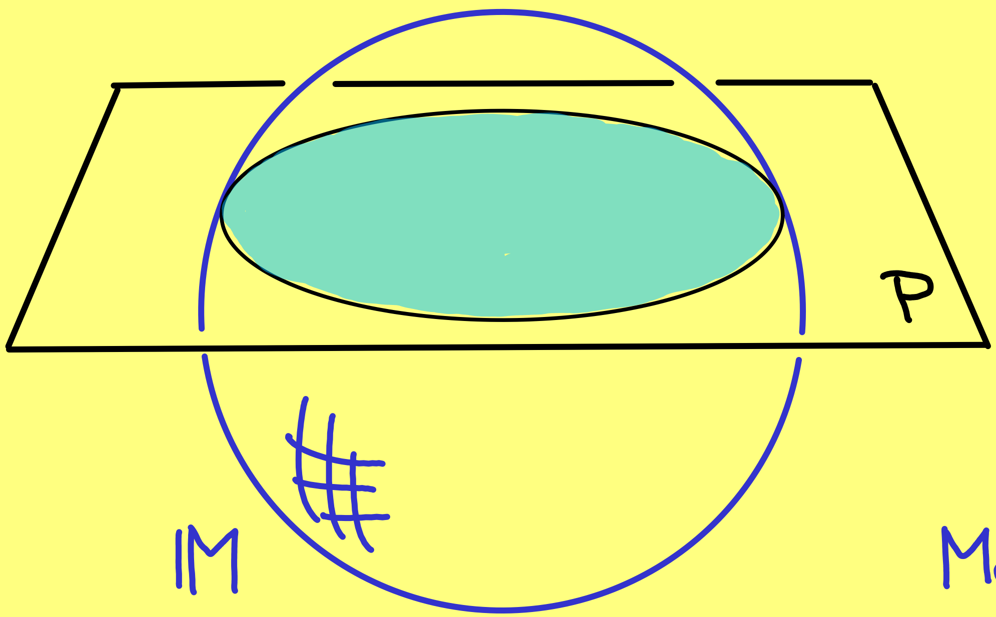
CONVERGENCE



solid body $\mathbb{M} \subseteq \mathbb{R}^3$

that is : compact and smoothly embed. $\partial\mathbb{M}$

Convergence



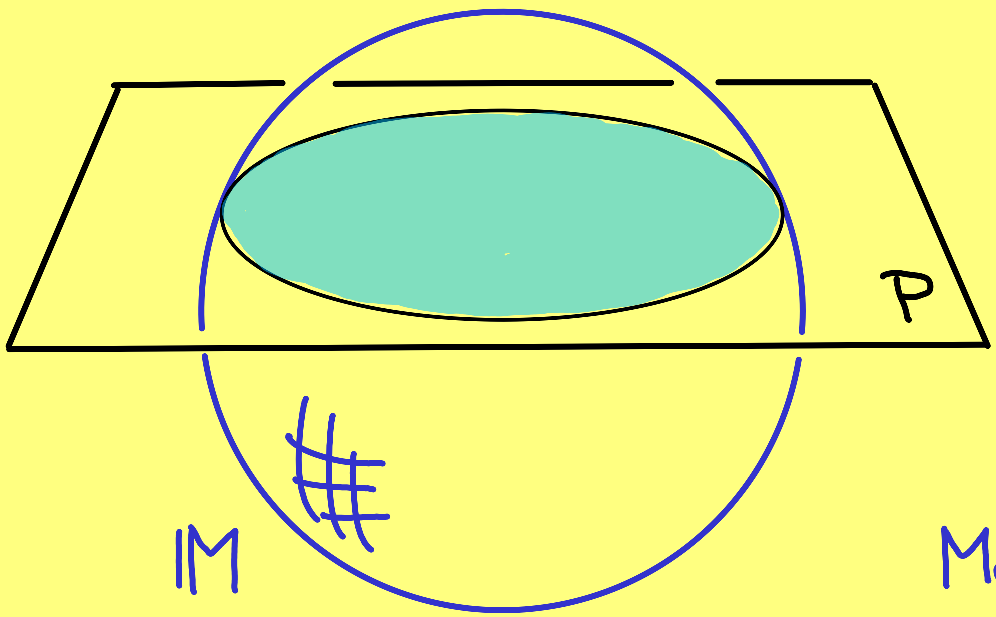
solid body $\mathbb{M} \subseteq \mathbb{R}^3$

that is: compact and

smoothly embed. $\partial\mathbb{M}$

$$\mathrm{Mean}(\mathbb{M}) = \int_{P \subseteq \mathbb{R}^3} \chi(\mathbb{M} \cap P)\, dP$$

Convergence



solid body $\mathbb{M} \subseteq \mathbb{R}^3$

that is: compact and

smoothly embed. $\partial\mathbb{M}$

$$\operatorname{Mean}(\mathbb{M}) = \int_{P \subseteq \mathbb{R}^3} \chi(\mathbb{M} \cap P)\, dP$$

THM. Solid body $\mathbb{M} \subseteq \mathbb{R}^n$, $1/\kappa_{max} \geq 4tn$, reach $> 2t\sqrt{n}$, C finite.

Then $\left| \operatorname{Mean}(\mathbb{M}) - \operatorname{Mean}(\mathbb{M}_t, t\sqrt{n}) \right| \leq c_{n-1,n} \cdot t\sqrt{n} \cdot C.$

HISTORY

Bottleneck stability: Cohen-Steiner, E, Harer 2006

Convergence: E, Pausinger 2013

THANK YOU