Goal-oriented space-time mesh adaptation

Savithru Jayasinghe, Yixuan Hu
David Darmofal

Aerospace Computational Design Laboratory
Massachusetts Institute of Technology

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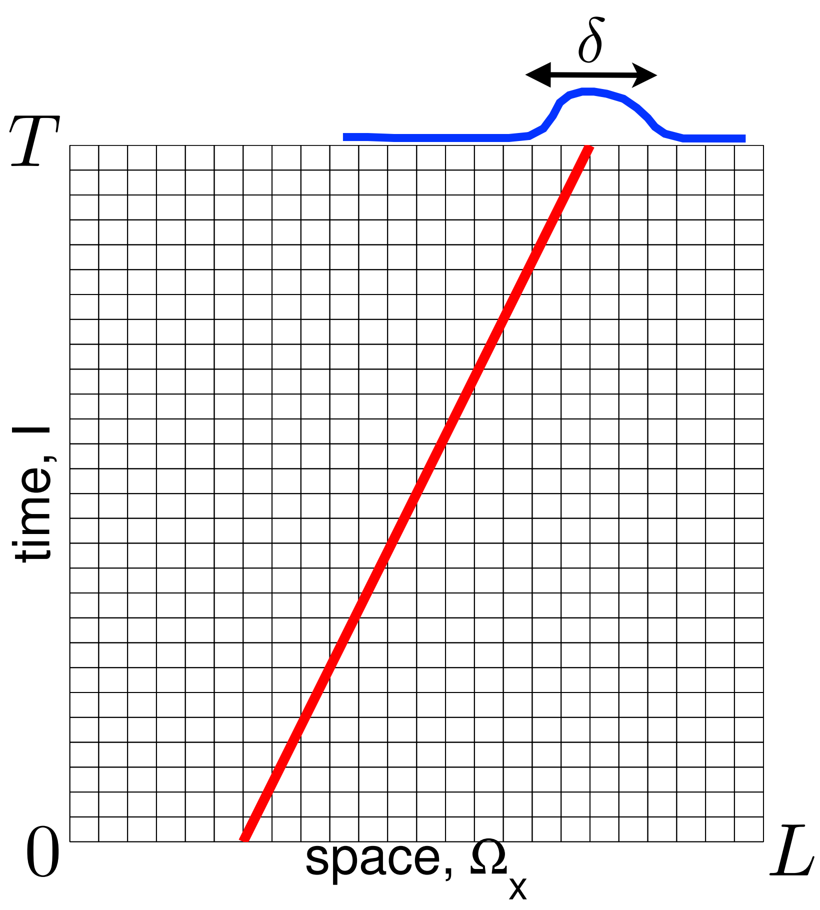
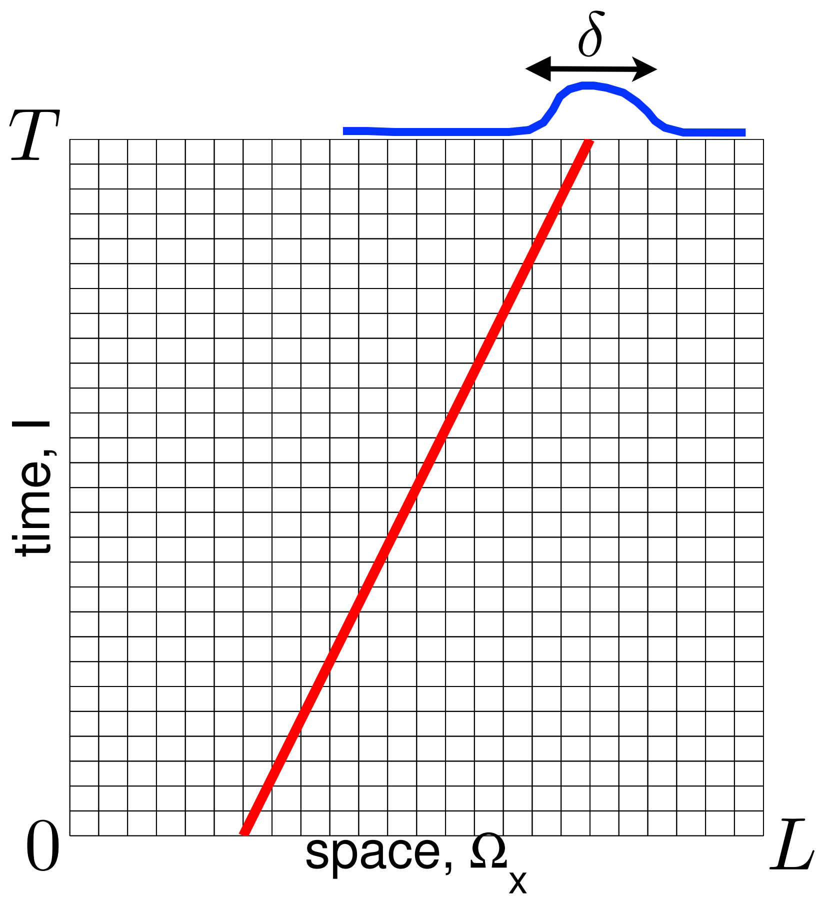
Space-time Adaptation



(a) uniform

$$N_{\text{dof}} = O\left((L/\delta)^2 \right)$$

Space-time Adaptation



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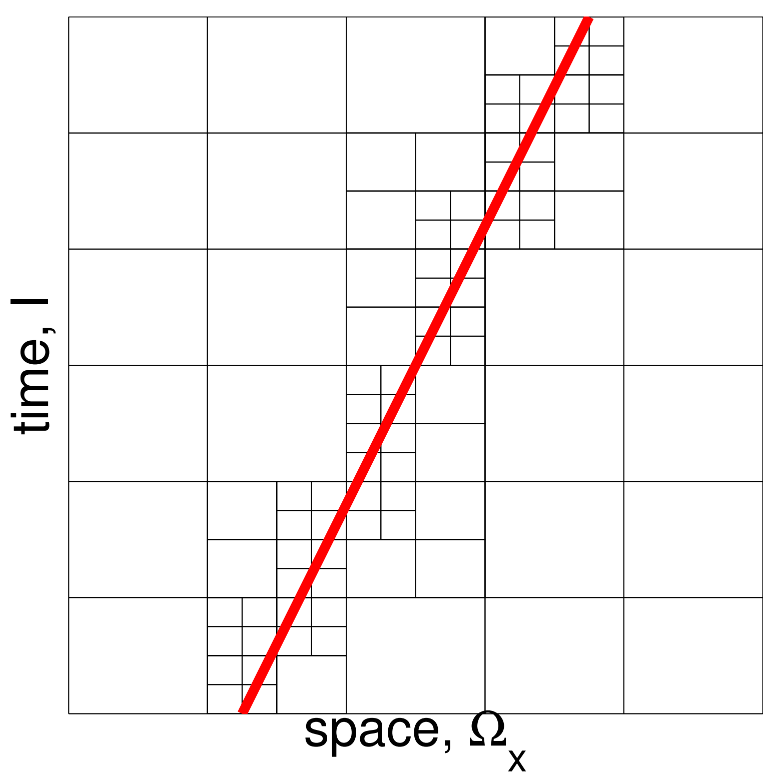
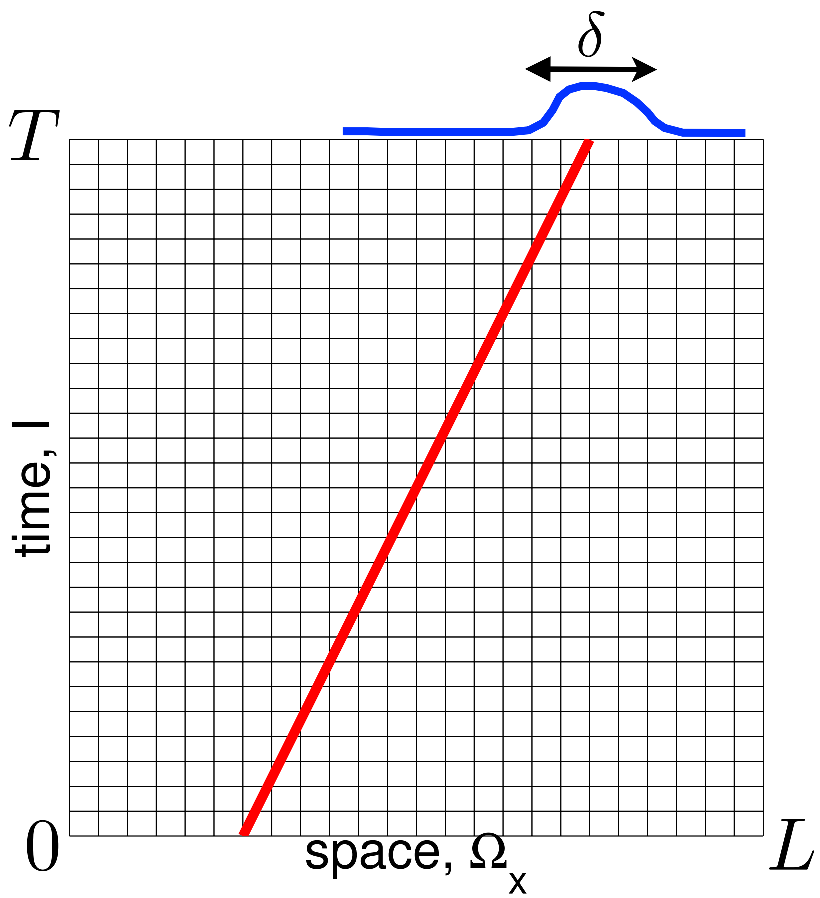


(b) space-time tensor-product

$$N_{\text{dof}} = O\big(\,(L/\delta)^2\,\big) \qquad N_{\text{dof}} = O(L/\delta\,)$$

Space-time Adaptation



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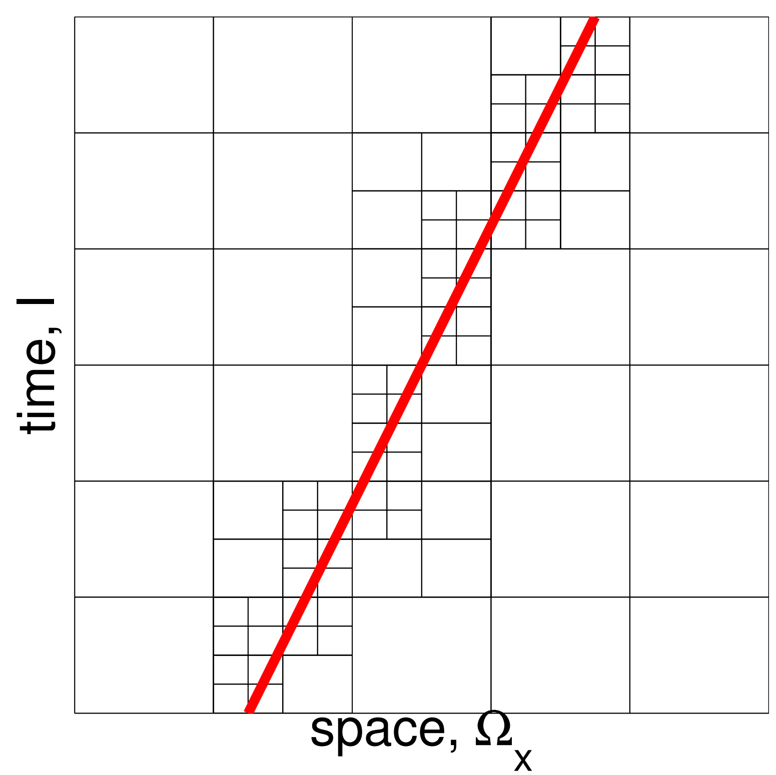
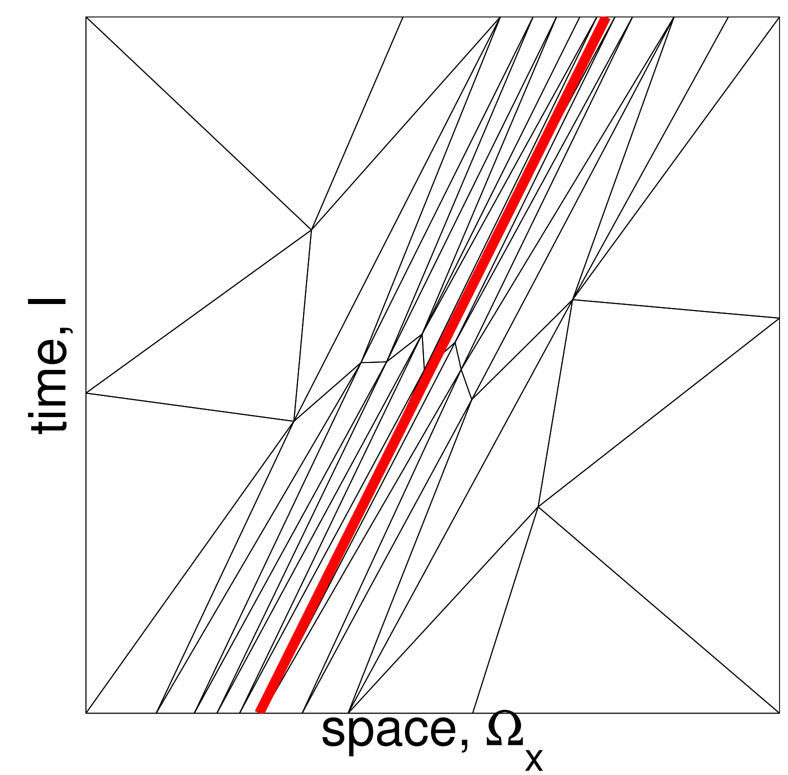


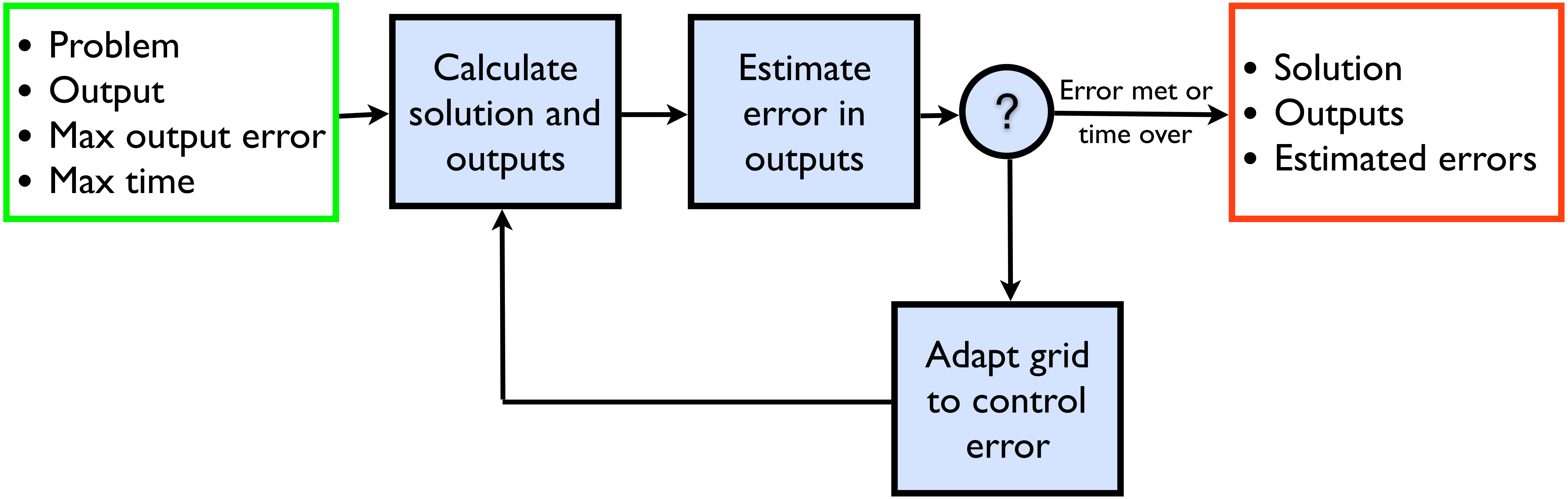
(b) space-time tensor-product



(c) space-time unstructured

$$N_{\mathrm{dof}} = O\Big((L/\delta)^2 \Big) \qquad N_{\mathrm{dof}} = O(L/\delta\,) \qquad N_{\mathrm{dof}} = O(1)$$

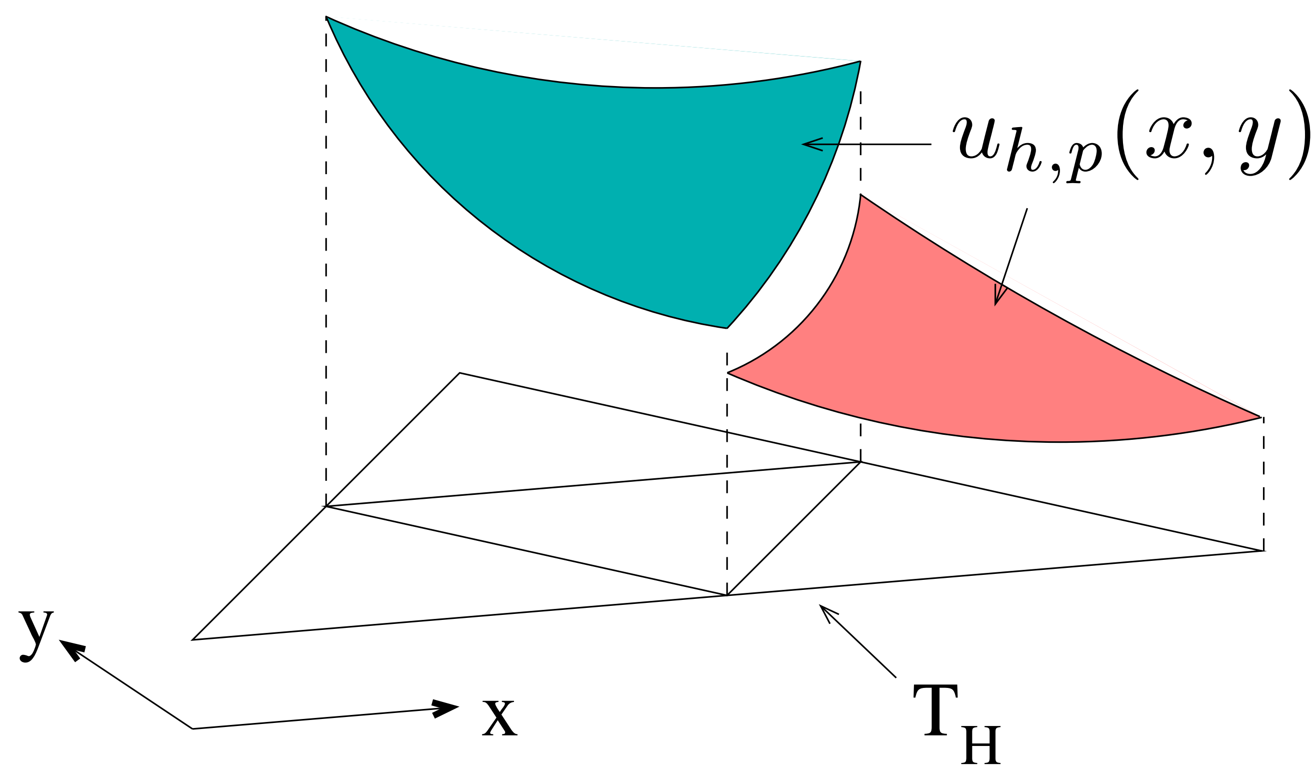
Goal-oriented adaptation



Objective: Increase reliability of simulation by estimating and autonomously controlling error in outputs

Discontinuous Galerkin method

- Approximations are degree p polynomials within elements but discontinuous between elements



DGFEM approximation: Find $u_{h,p} \in \mathcal{V}_{h,p}$ such that

$$\mathcal{R}_{h,p}(u_{h,p}, v_{h,p}) = 0, \qquad \forall v_{h,p} \in \mathcal{V}_{h,p}$$

Output error estimation

(Becker & Rannacher, 2001)

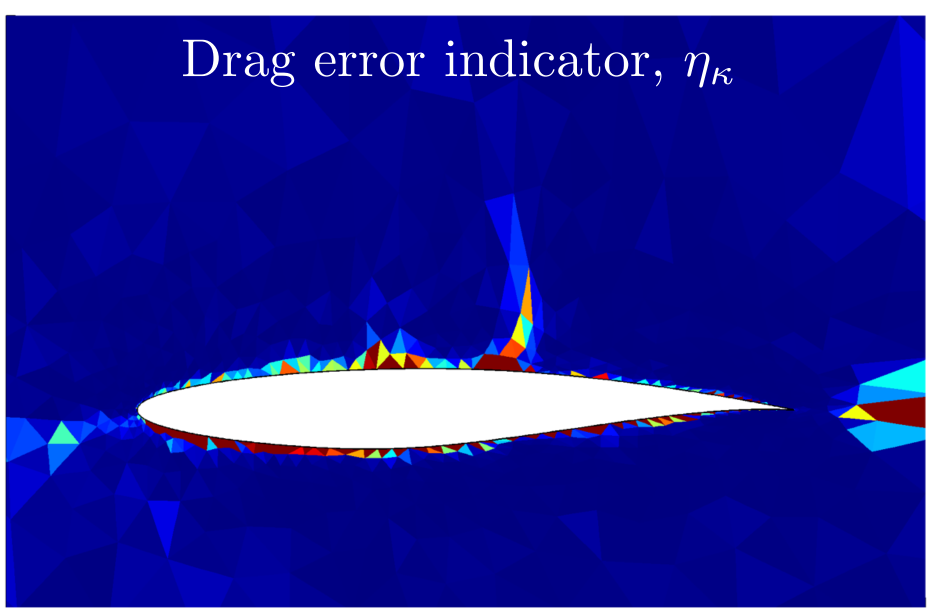
$$J_{h,p} - J \approx \mathcal{E}_{h,p'} \equiv -\mathcal{R}_{h,p}(u_{h,p}, \psi_{h,p'}),$$

where $\psi_{h,p'}$ is the adjoint in $p' = p + 1$.

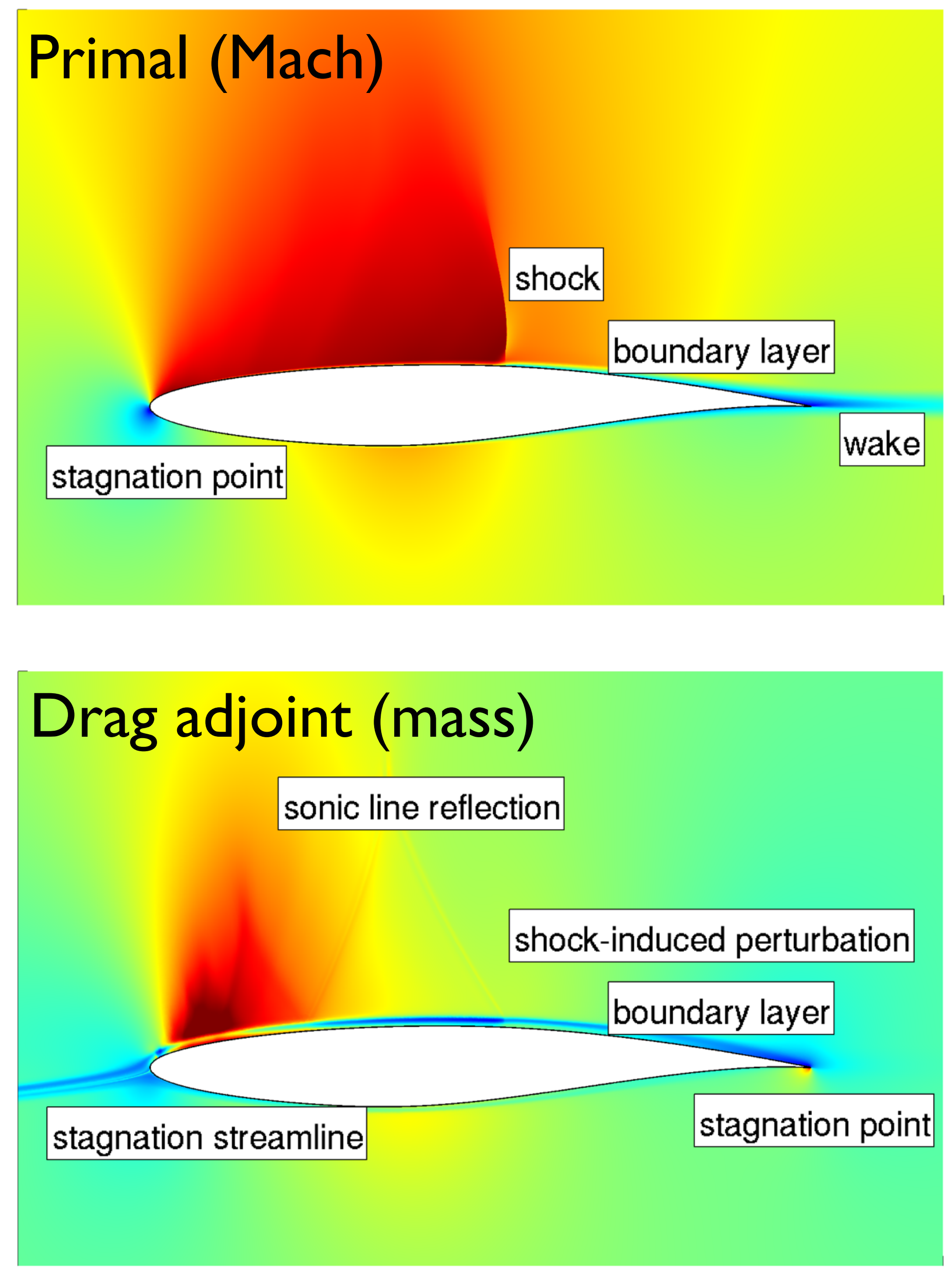
Localized to elemental error indicator,

$$\eta_\kappa \equiv \left| \mathcal{R}_{h,p}(u_{h,p}, \psi_{h,p'}|_\kappa) \right|$$

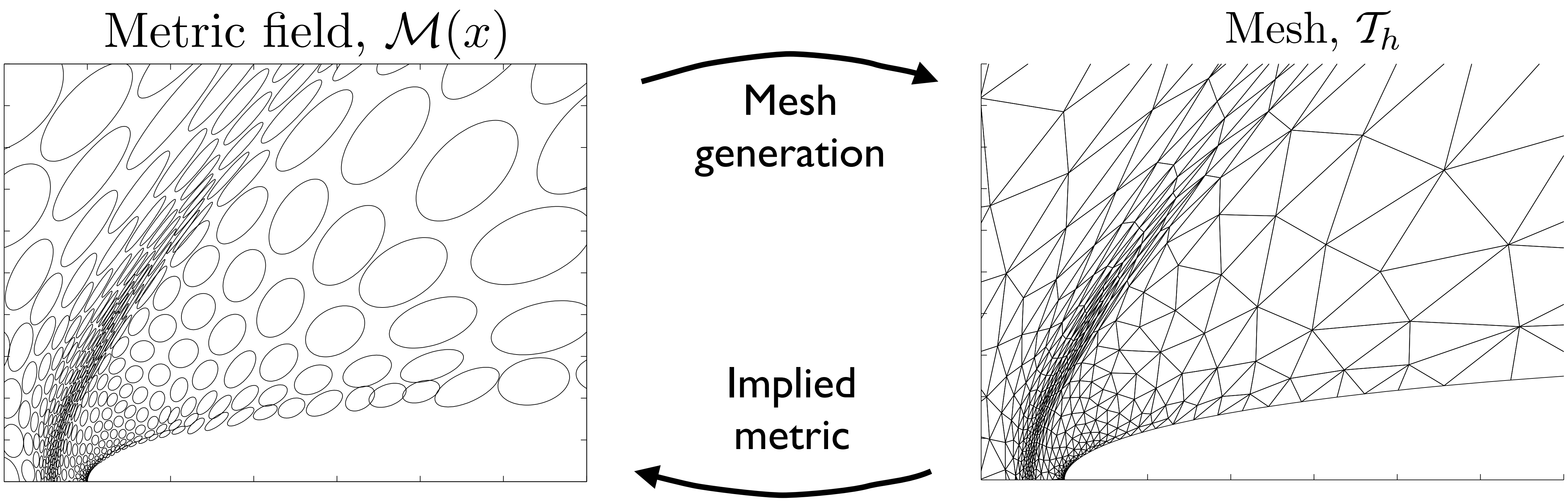
Dual-weighted Residual (DWR) error indicator



Transonic RANS example



Continuous Optimization: Mesh-metric Duality



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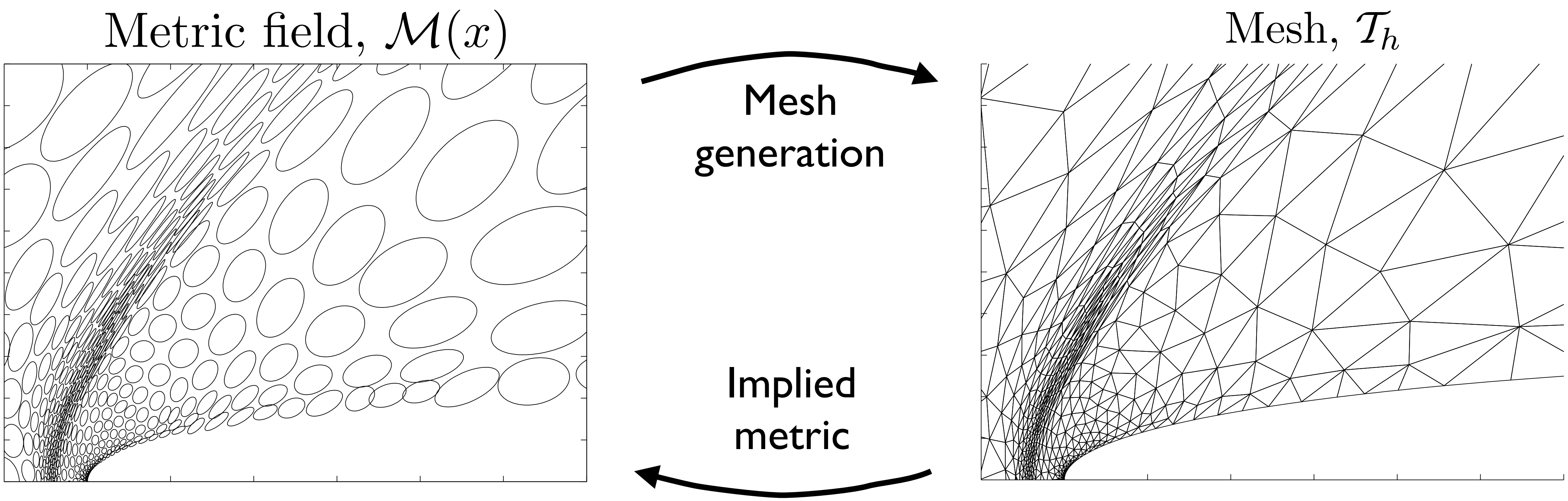

- (Intractable) discrete optimization problem

$$\mathcal{T}_h^* = \arg\inf_{\mathcal{T}_h} \mathcal{E}(\mathcal{T}_h) \quad \text{s.t.} \quad \mathcal{C}(\mathcal{T}_h) = \text{Cost}$$

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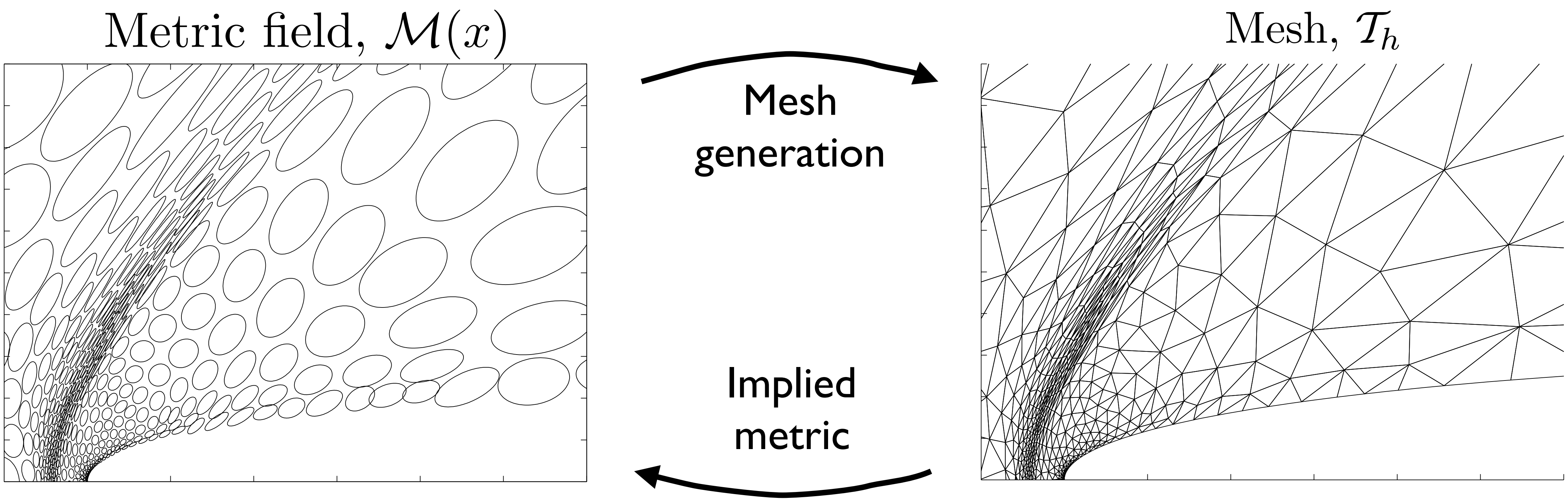

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$$\mathcal{T}_h^* = \arg\inf_{\mathcal{T}_h} \mathcal{E}(\mathcal{T}_h) \quad \text{s.t.} \quad \mathcal{C}(\mathcal{T}_h) = \text{Cost}$$

- Continuous relaxation (Loseille & Alauzet 2011)

$$\mathcal{M}^* = \arg\inf_{\mathcal{M}} \mathcal{E}(\mathcal{M}) \quad \text{s.t.} \quad \mathcal{C}(\mathcal{M}) = \text{Cost}$$

MOESS Algorithm

(Mesh Optimization via Error Sampling & Synthesis)

Yano & Darmofal, 2012

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- Solve primal and adjoint on current grid

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- Optimize metric to reduce error

Yano & Darmofal, 2012

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MOESS Algorithm

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- Solve primal and adjoint on current grid
- Determine error-metric model via local sampling
- Optimize metric to reduce error
- Remesh using improved metric

MOESS Locality Assumptions

- Assume locality of error and cost functionals

$$\mathcal{E}(\mathcal{M}) = \int_\Omega e(\mathcal{M}(x))dx \quad \text{and} \quad \mathcal{C}(\mathcal{M}) = \int_\Omega c(\mathcal{M}(x))dx$$

- Relating error function to error indicator

$$\mathcal{E}(\mathcal{M}) = \int_\Omega e(\mathcal{M}(x))dx = \sum_\kappa \eta_\kappa(\mathcal{M}_\kappa)$$

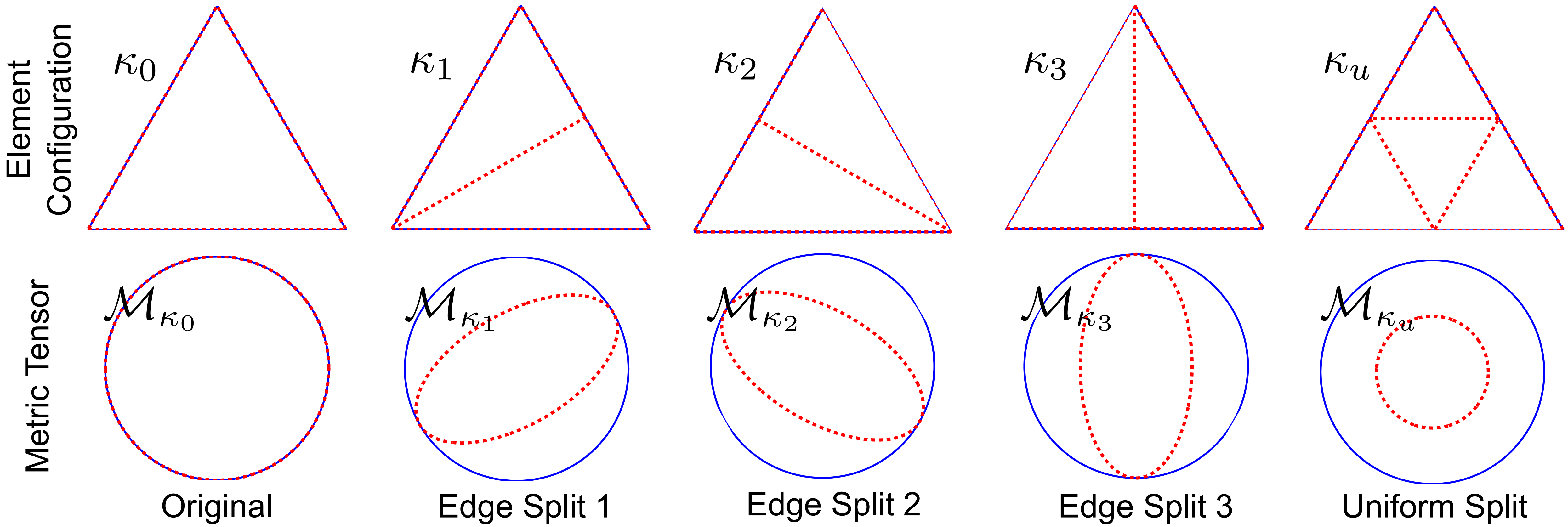
- For remainder of this work, choose cost to be DOF:

$$\mathcal{C}(\mathcal{M}) = \int_\Omega c(\mathcal{M}(x))dx = \sum_\kappa \rho_\kappa(\mathcal{M}_\kappa)$$

where ρ_κ are the DOF in region κ.

Yano & Darmofal, 2012

Local sampling



- For each configuration, solve local problems keeping states outside of κ_0 fixed
- Determine error estimate $\eta_{\kappa_i} = R_{h,p}(u_{h,p}^{\kappa_i}, \psi_{h,p+1}|_{\kappa_0})$
- Produces a set of pairs, $\{\mathcal{M}_{\kappa_i}, \eta_{\kappa_i}\}$
- Using affine-invariant metric description, produce model for $\log \eta_\kappa(\mathcal{M}\kappa)$

Yano & Darmofal, 2012

Affine-invariant metric framework

- Employ affine-invariant description of a metric space (Pennec et al, 2006)

$$S_\kappa = \log\left(\mathcal{M}_{\kappa_0}^{-1/2}\mathcal{M}_\kappa\mathcal{M}_{\kappa_0}^{-1/2}\right)$$

- S_κ (the step matrix) can be decomposed into $S_\kappa = s_\kappa I + \tilde{S}_\kappa$
- s_κ is isotropic and controls the area change
- $\tilde{S}_\kappa$ controls orientation and stretching changes
- First-order optimality conditions become

$$\begin{aligned} \frac{\partial \eta_\kappa}{\partial s_\kappa} - \lambda \frac{\partial \rho_\kappa}{\partial s_\kappa} &= 0 \\ \frac{\partial \eta_\kappa}{\partial \tilde{S}_\kappa} &= 0 \end{aligned}$$

Error model synthesis

- Define logarithmic error model $f_{\kappa_i} \equiv \log(\eta_{\kappa_i}/\eta_{\kappa_0})$

$$\{\mathcal{M}_{\kappa_i}, \eta_{\kappa_i}\} \to \{S_{\kappa_i}, f_{\kappa_i}\}$$

- Perform a least-squares fit to synthesis $f_\kappa(S_\kappa) = \text{tr}(R_\kappa S_\kappa)$:

$$R_\kappa = \arg\min_{Q \in Sym_d} \sum_{i=1}^{n_{\text{config}}} (f_{\kappa_i} - \text{tr}(Q S_{\kappa_i}))^2$$

- This gives $\eta_\kappa(S_\kappa) = \eta_{\kappa_0} \exp(r_\kappa s_\kappa d) \exp\left(\text{tr}\left(\tilde{R}_\kappa \tilde{S}_\kappa\right)\right)$
- For isotropic error and meshing this model reduces to,

$$\eta_\kappa^{\text{iso}}(h) = \eta_{\kappa_0} \left(\frac{h}{h_0}\right)^{r_\kappa^{\text{iso}}}$$

Yano & Darmofal, 2012

MOESS Properties

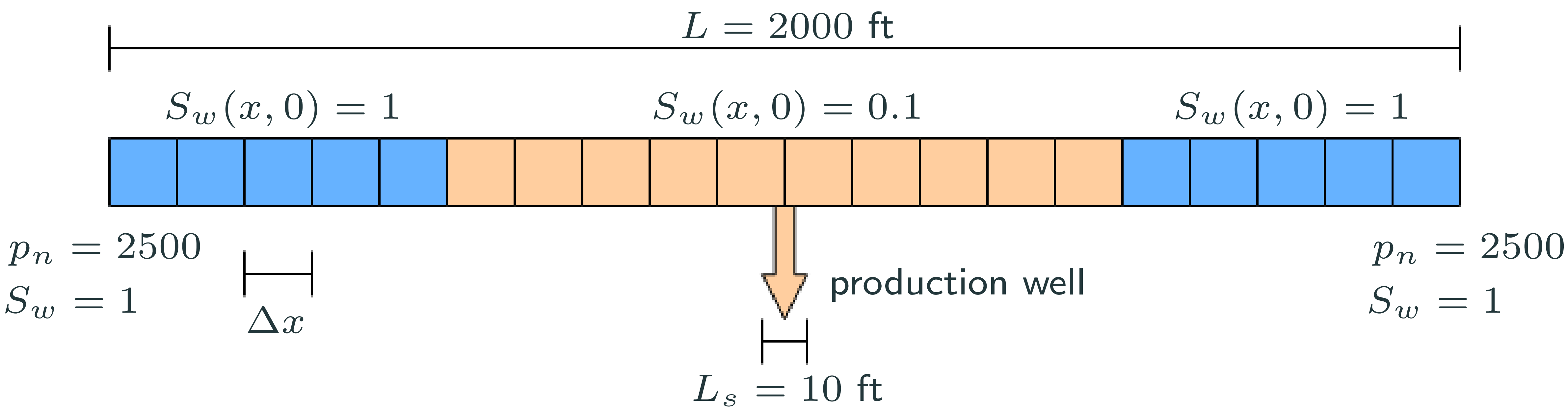
(Mesh Optimization via Error Sampling & Synthesis)

- Applicable to any discretization order

- Anisotropy adaptation driven entirely by error estimate

- No *a priori* assumptions on convergence rates

Application:
Two-phase porous media flow



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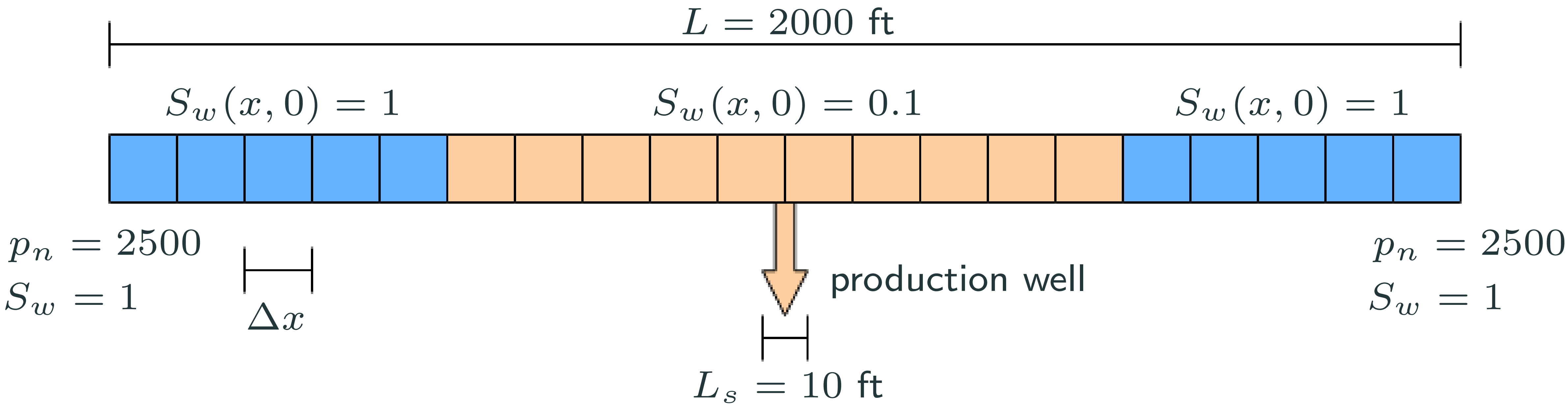

$$\frac{\partial}{\partial t}(\rho_w \phi S_w) - \frac{\partial}{\partial x}\left(\rho_w K \frac{k_{rw}}{\mu_w}\frac{\partial p_w}{\partial x}\right) = \rho_w q_w$$

$$\frac{\partial}{\partial t}(\rho_n \phi S_n) - \frac{\partial}{\partial x}\left(\rho_n K \frac{k_{rn}}{\mu_n}\frac{\partial p_n}{\partial x}\right) = \rho_n q_n$$

Closure relations:

$$S_w + S_n = 1;\ p_c(S_w) = p_n - p_w;\ \rho_\alpha = \rho_{\alpha_{\text{ref}}} e^{c_\alpha (p_\alpha - p_{\text{ref}})};\ k_{r\alpha} = S_\alpha^2;\ \text{etc.}$$

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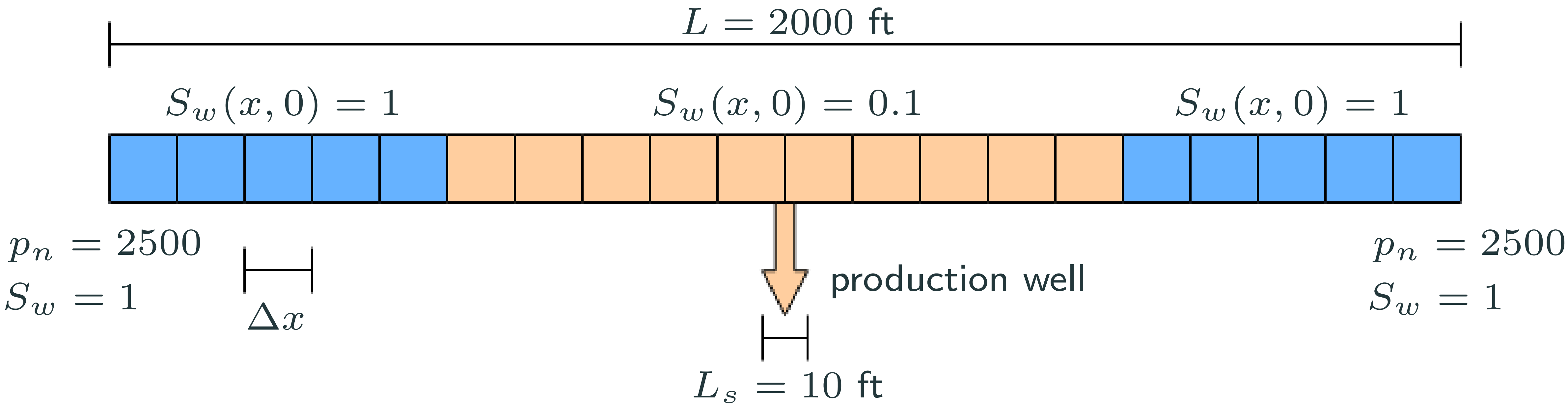

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$$\text{Output: } J = \frac{\text{Volume oil extracted}}{\text{Initial oil volume}}$$

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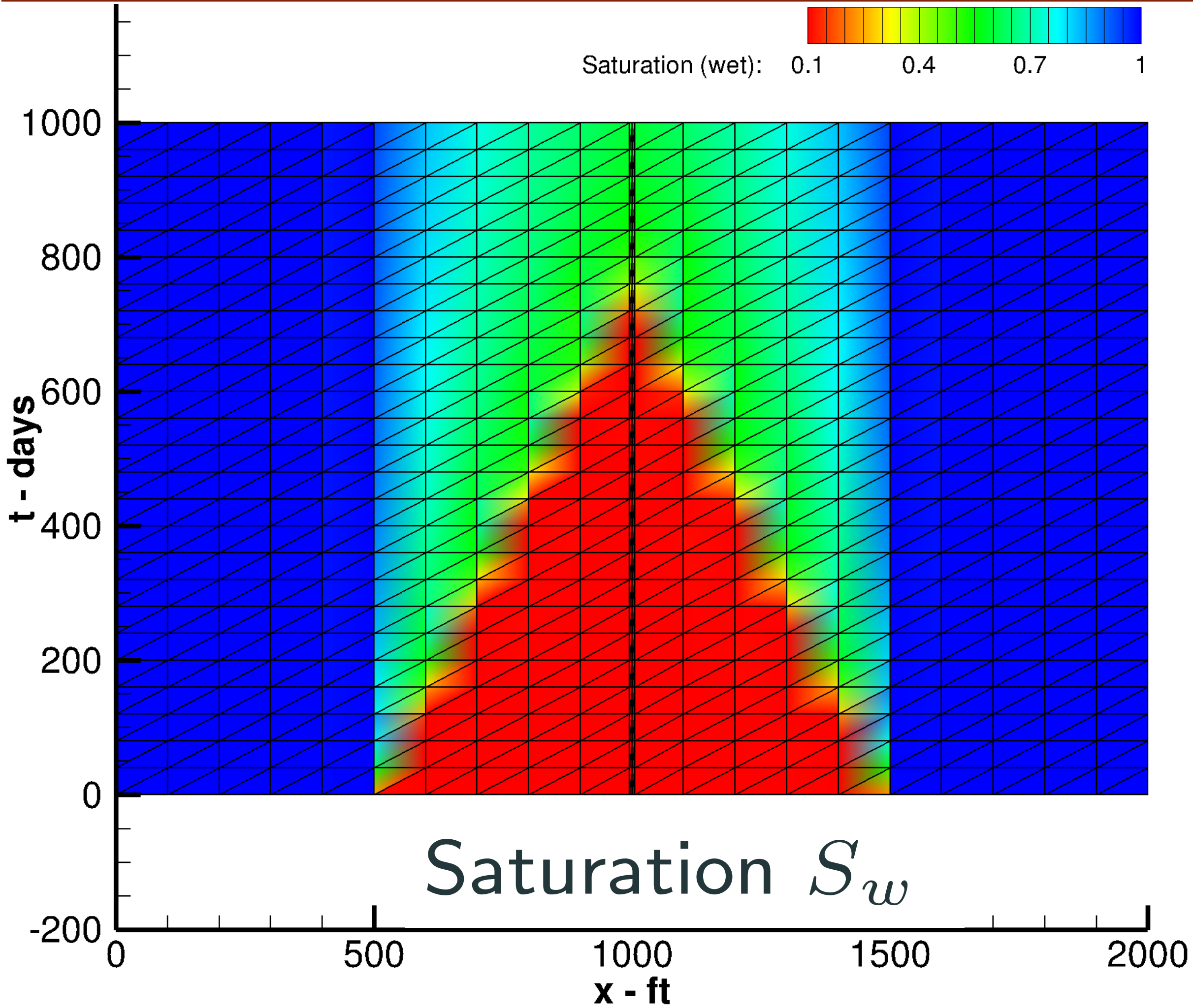
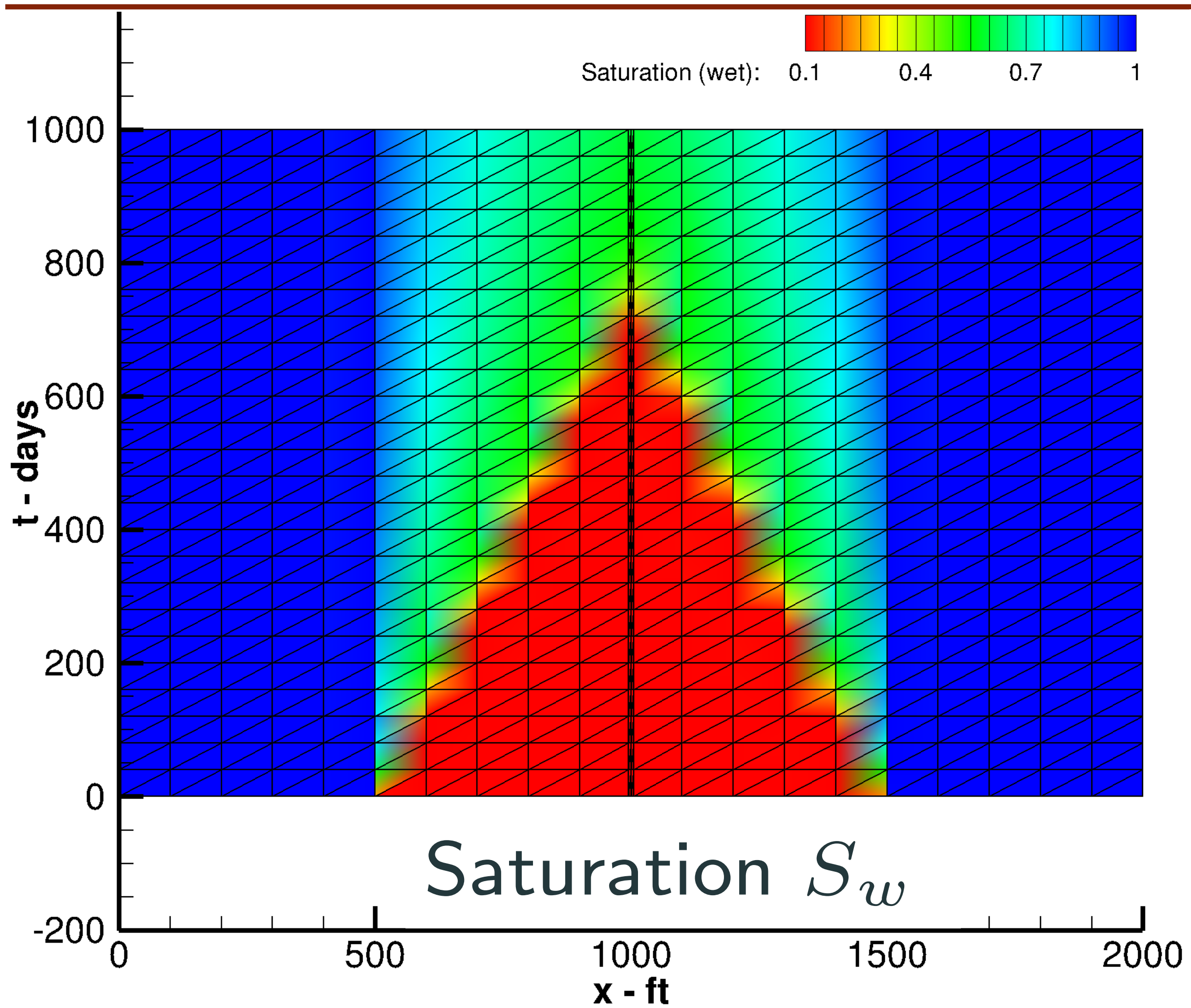
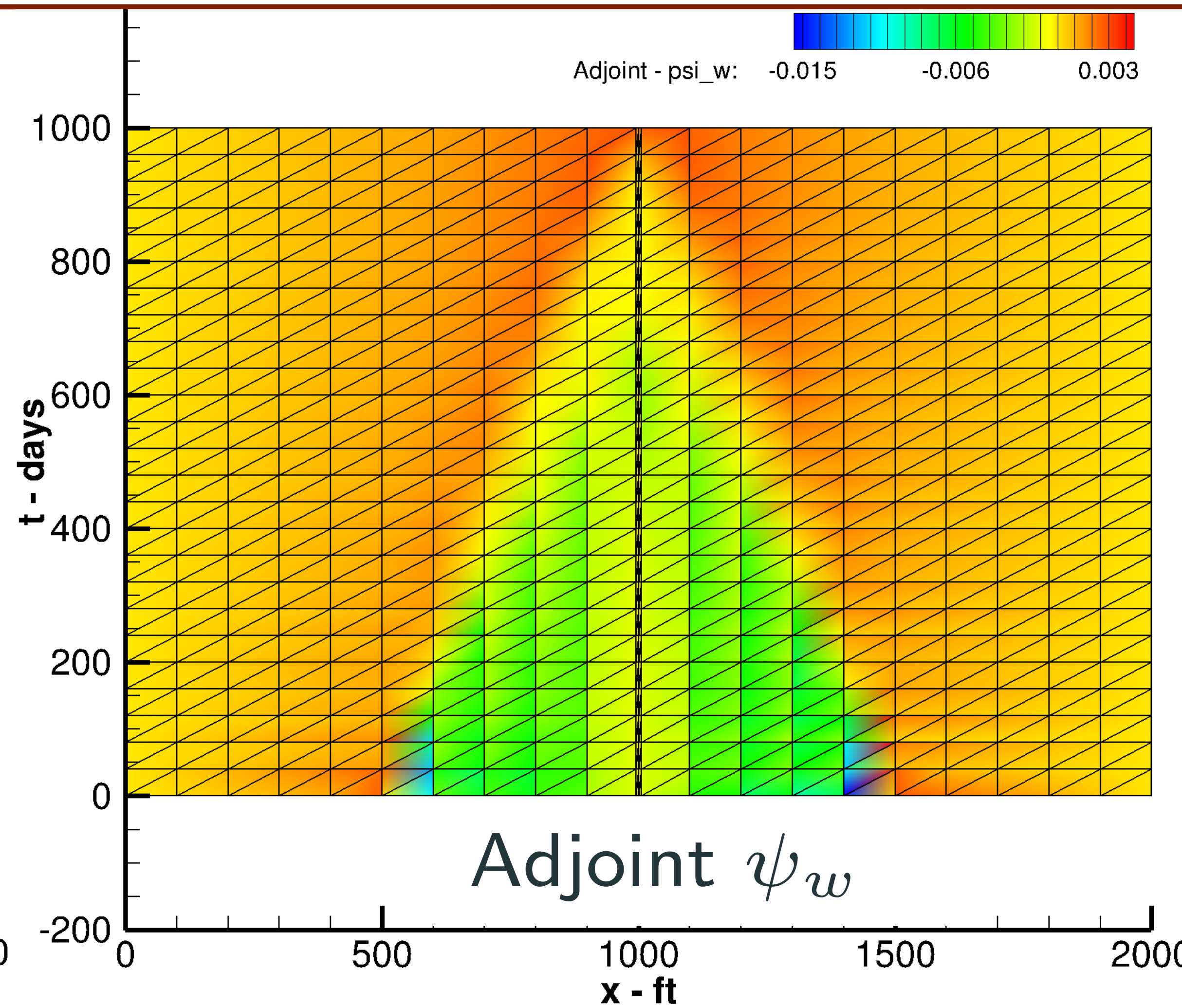


p=1

25K DOF

Application: Two-phase porous media flow





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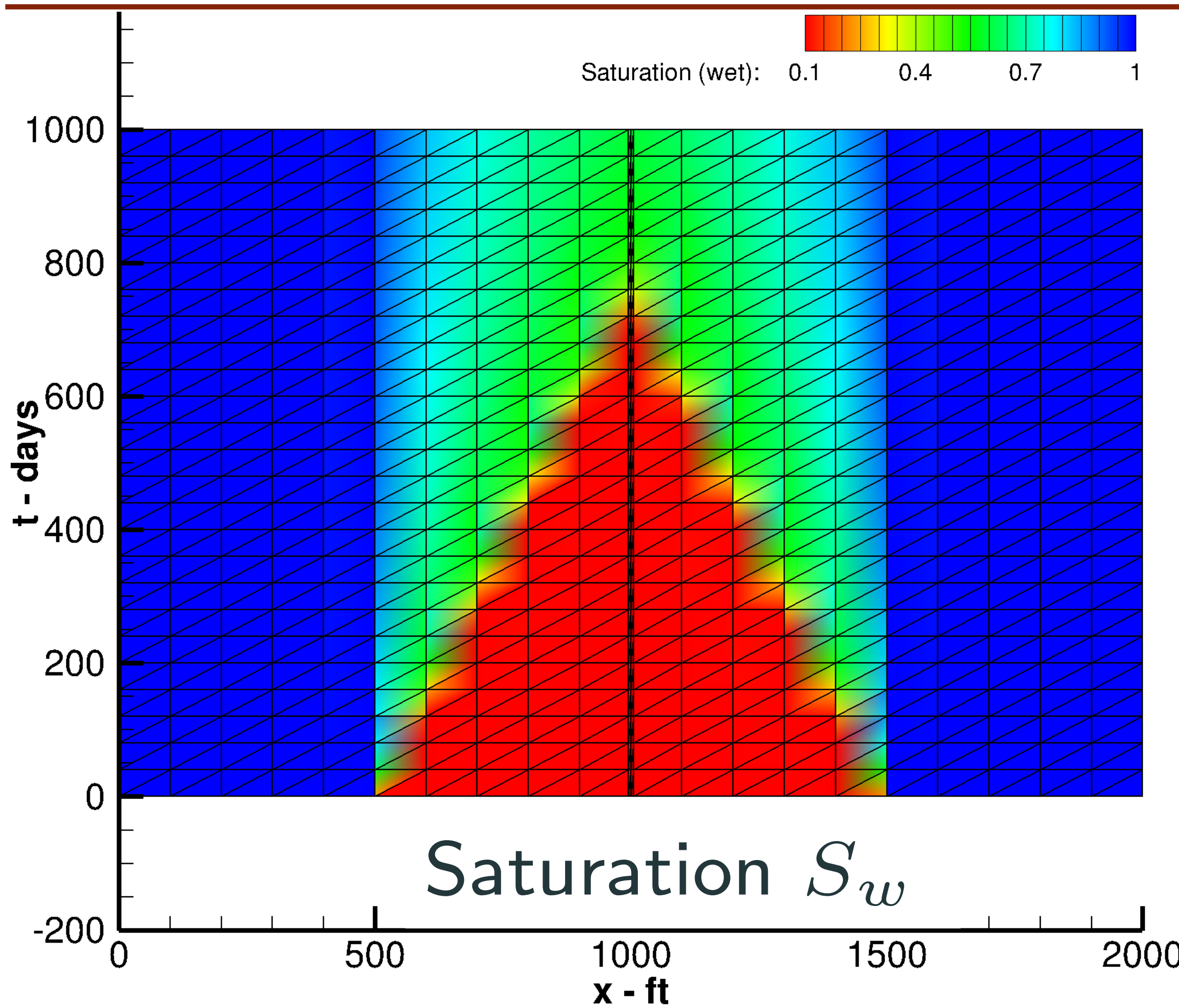

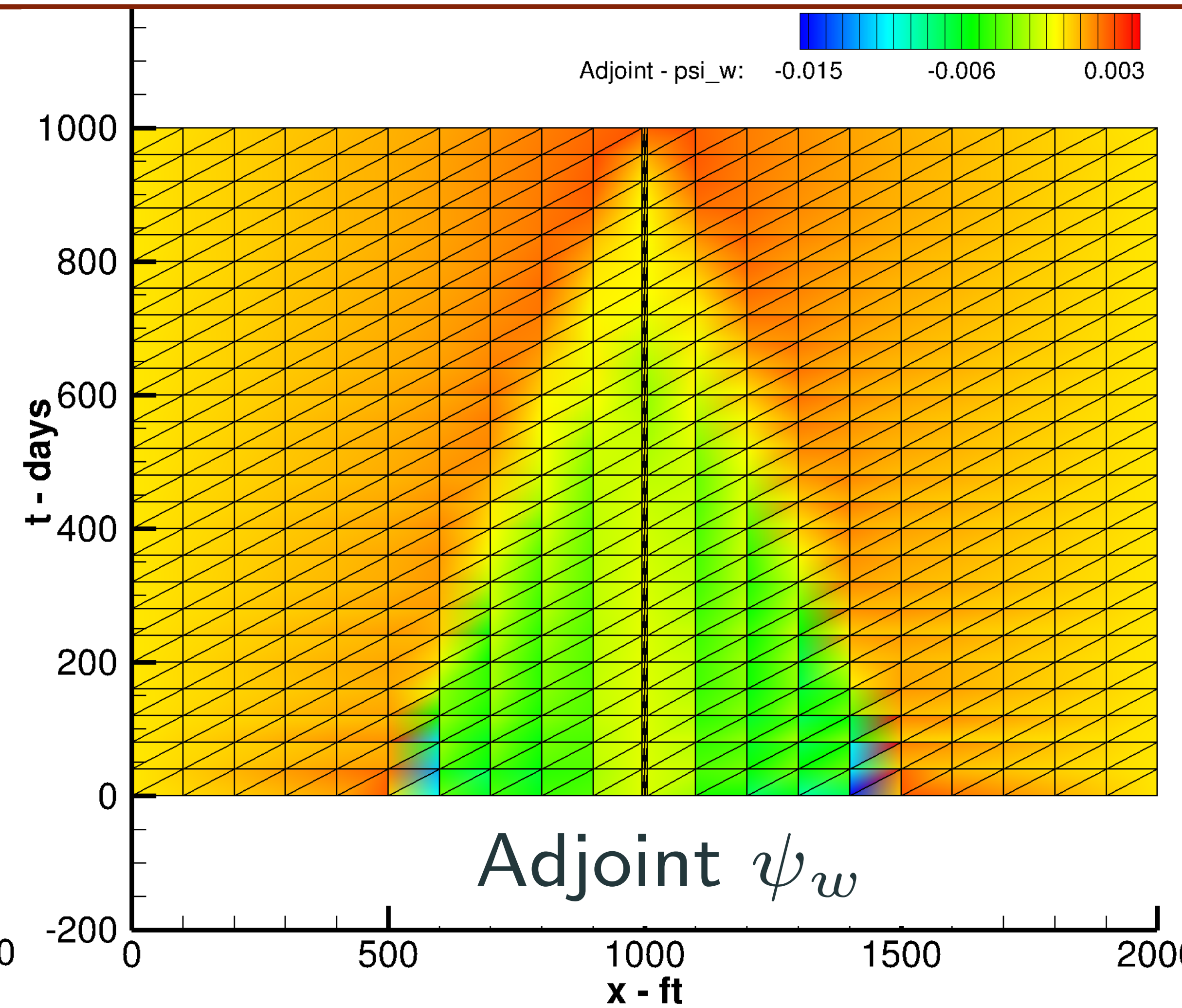

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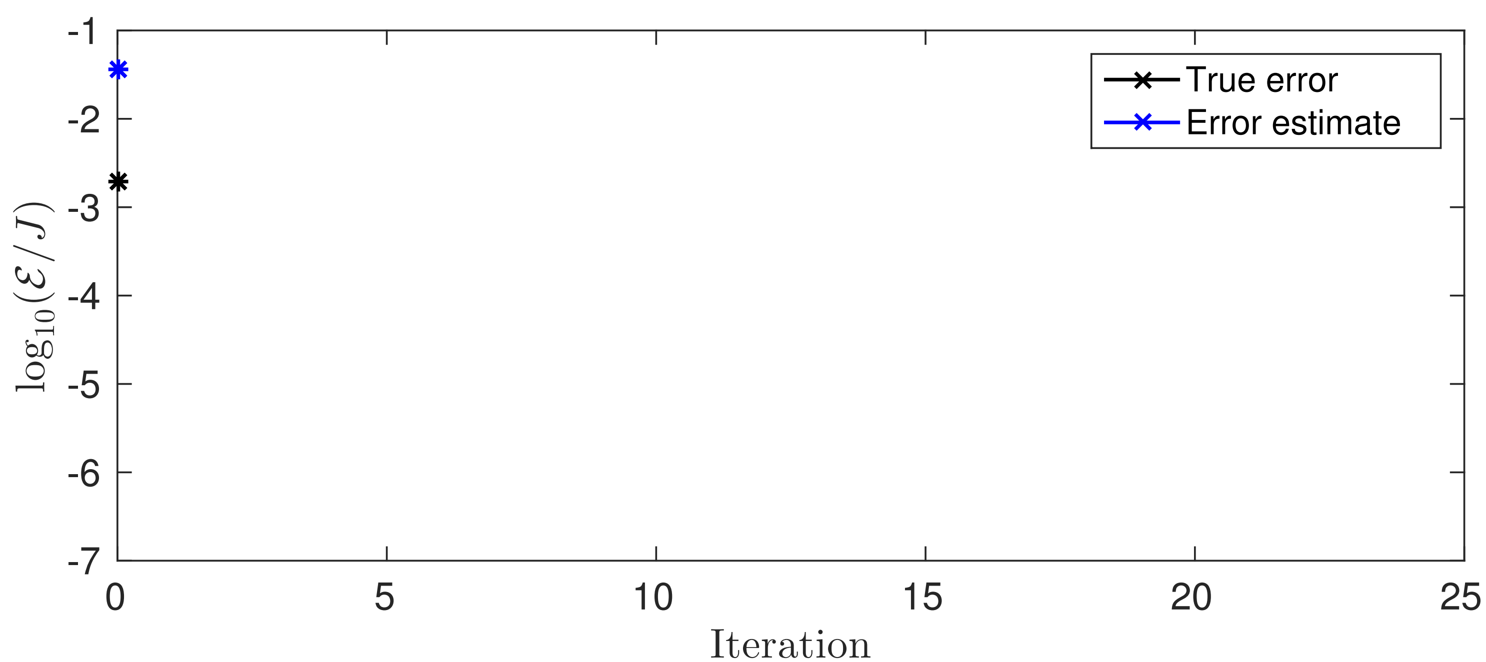

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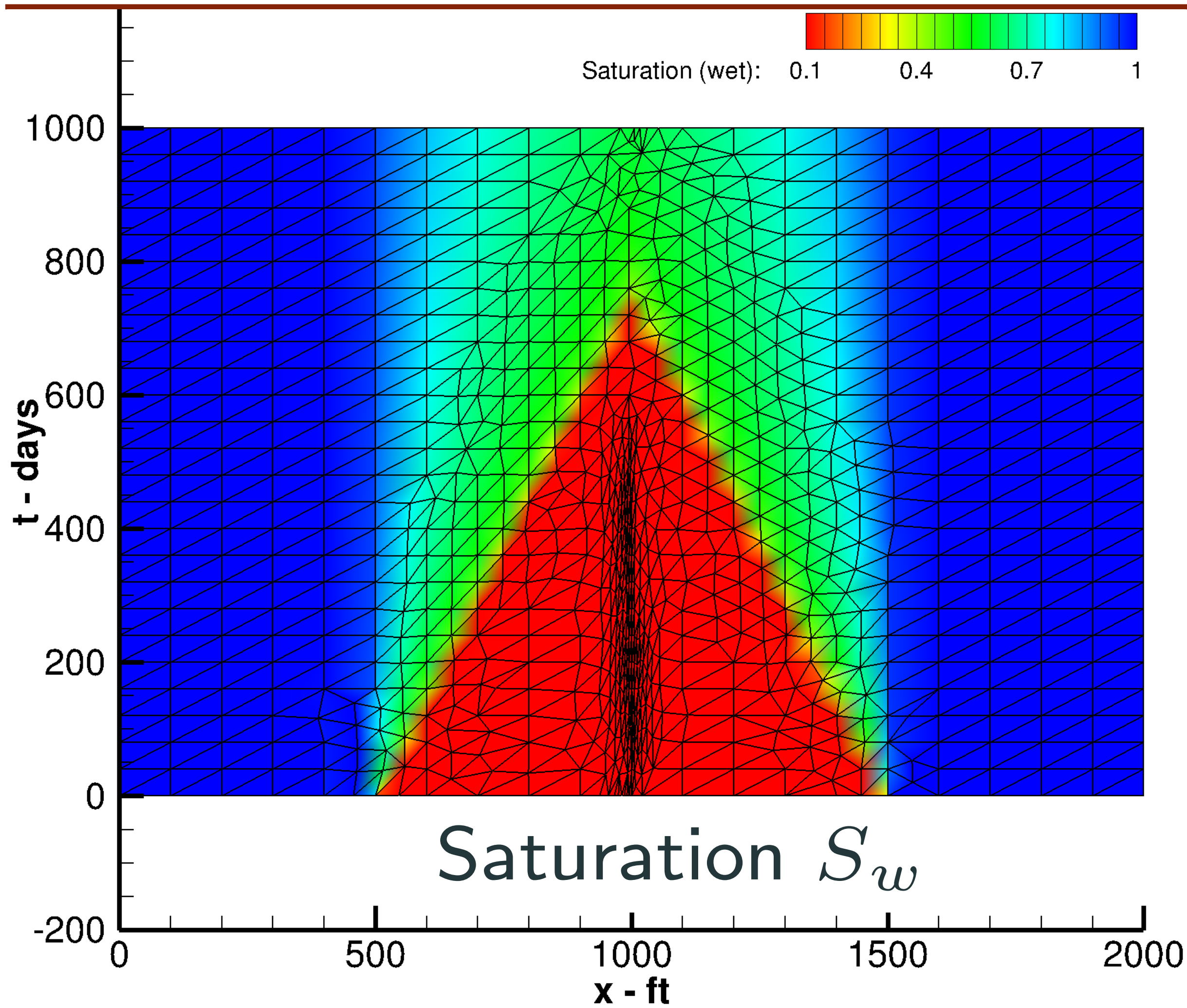

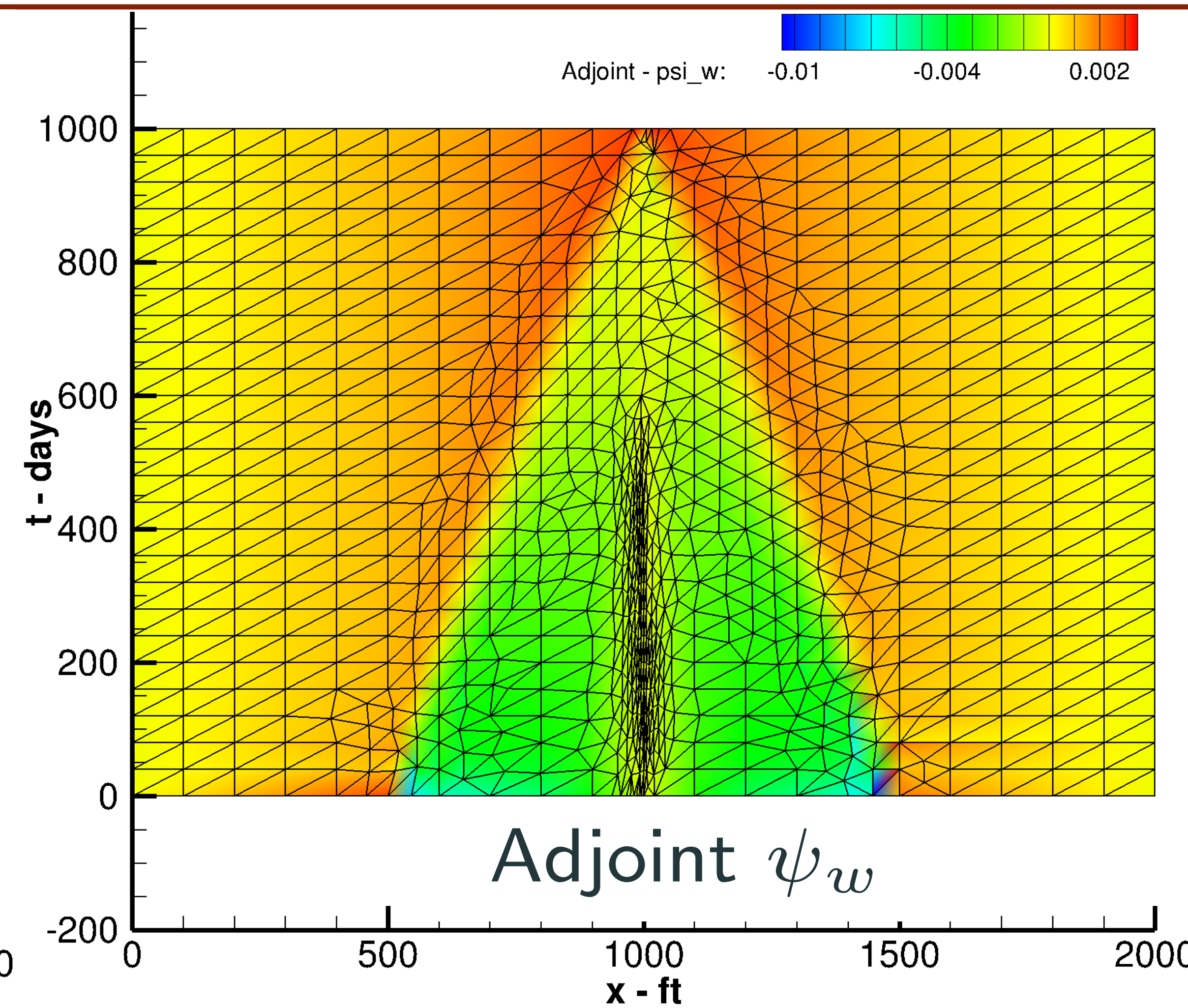

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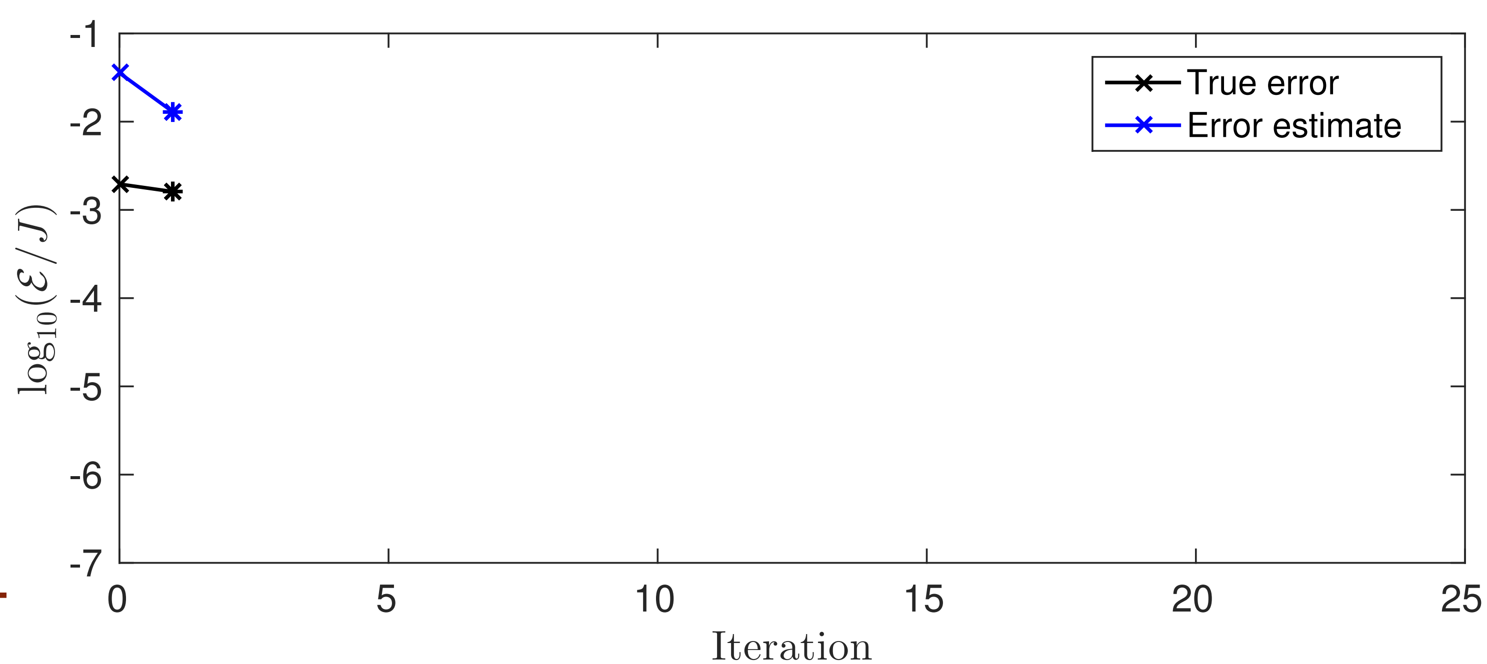

Adaptive meshes generated with feflo.a (Loseille)

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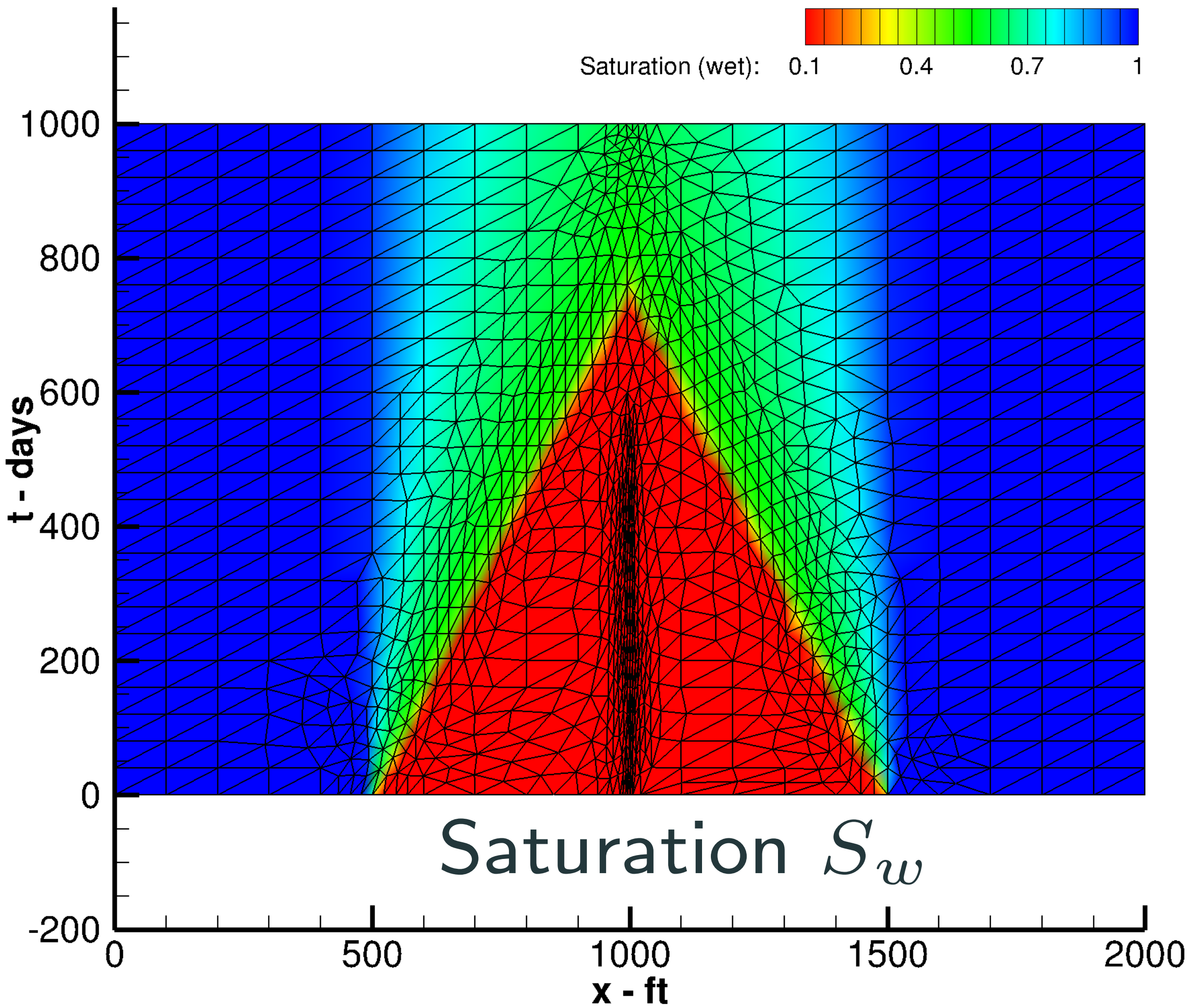

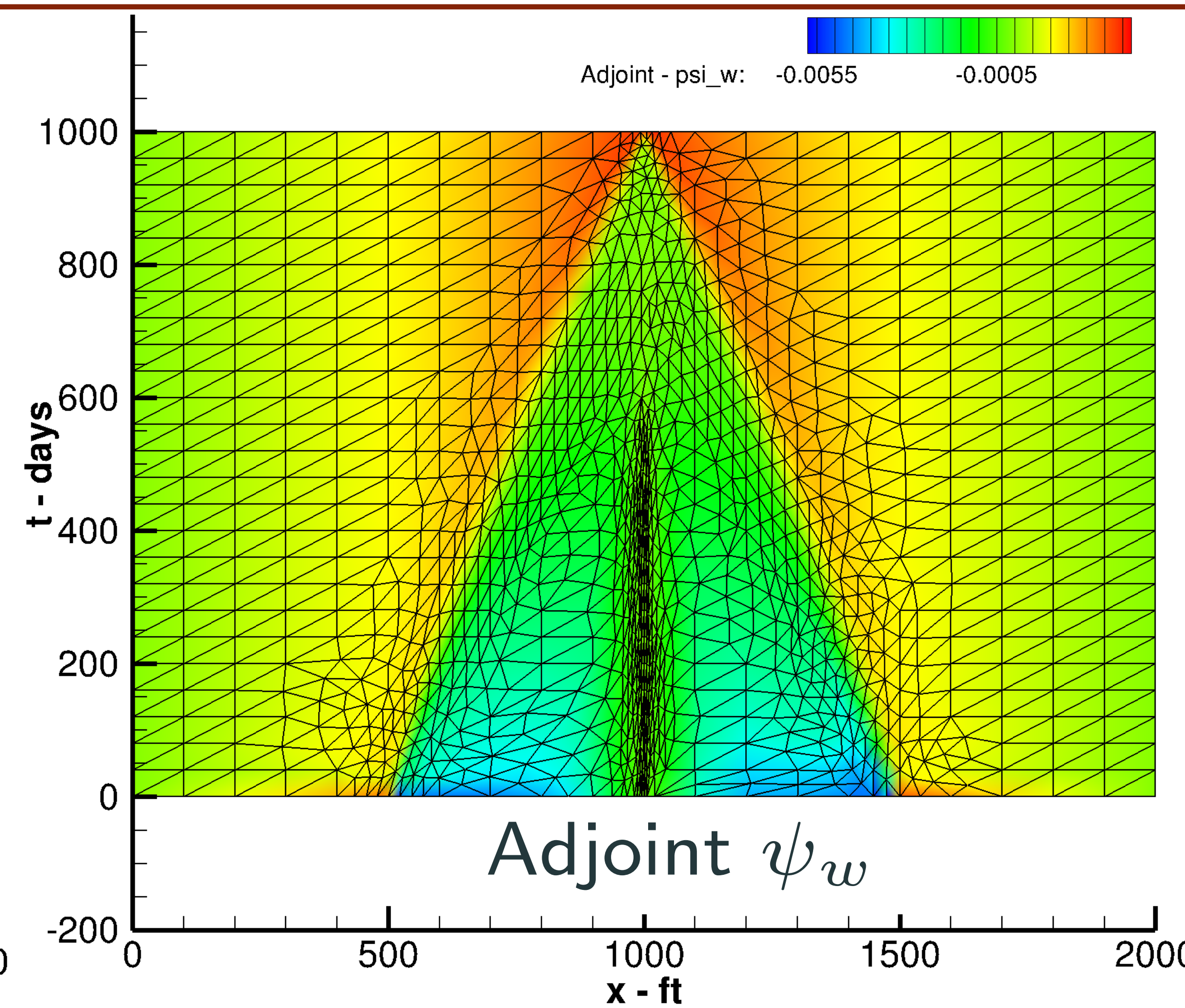

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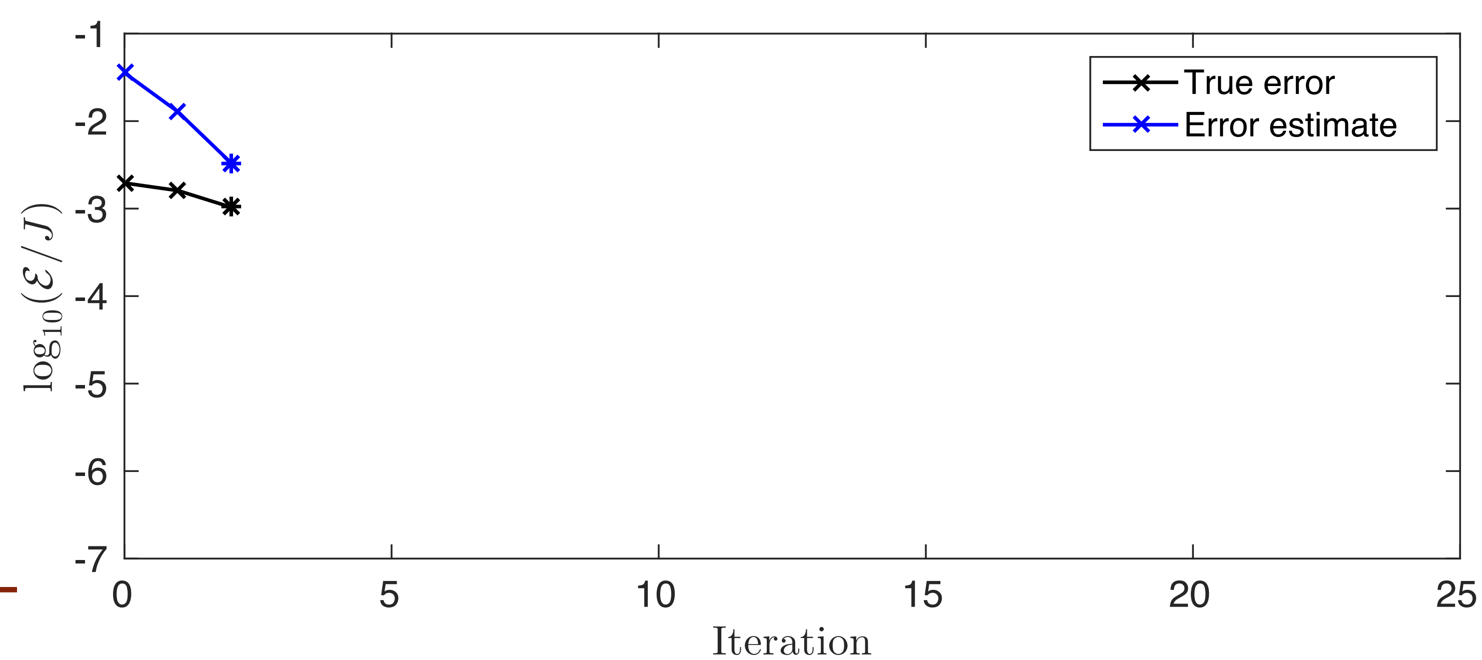

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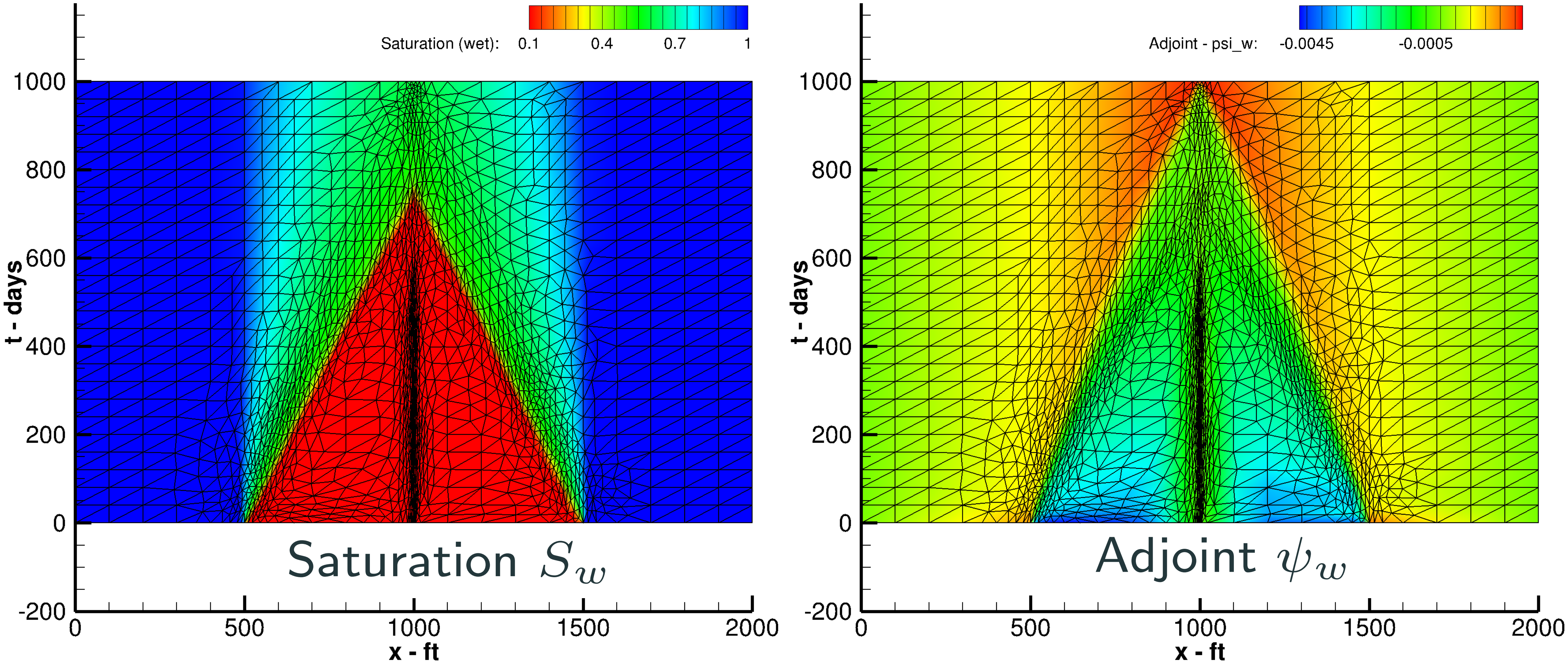

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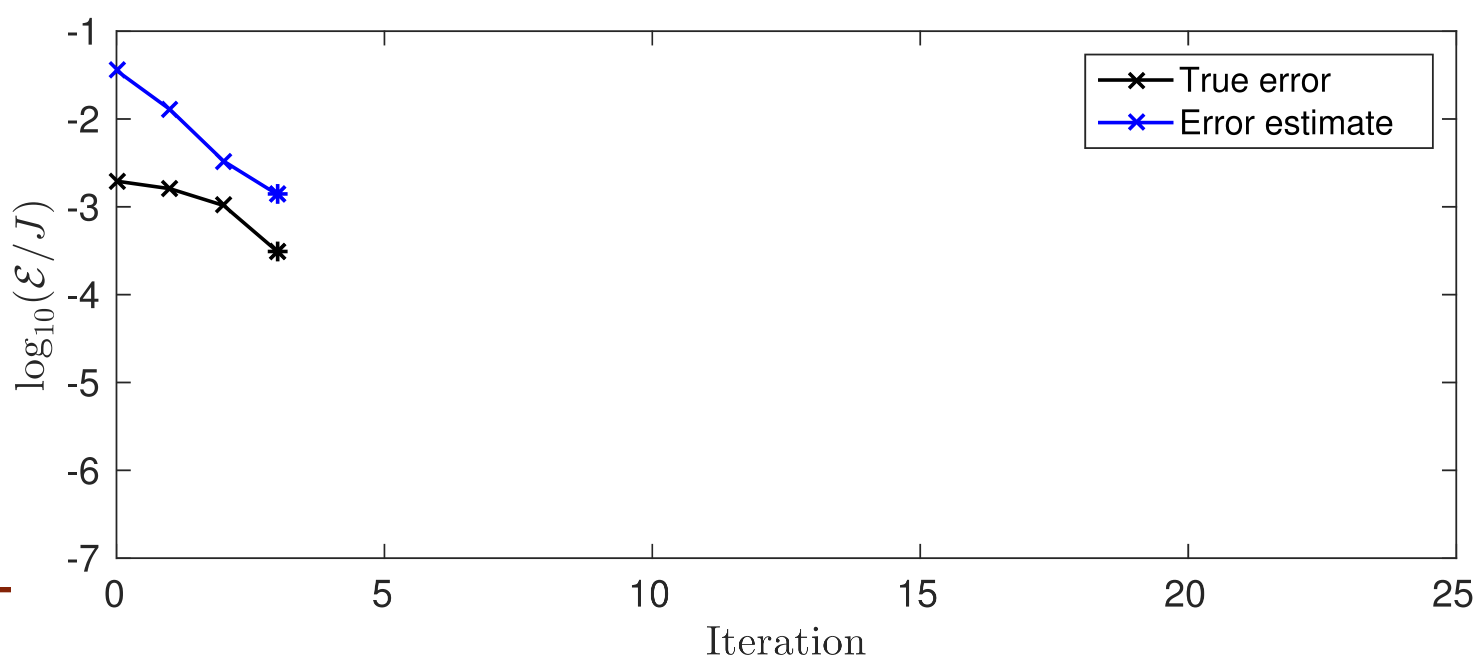

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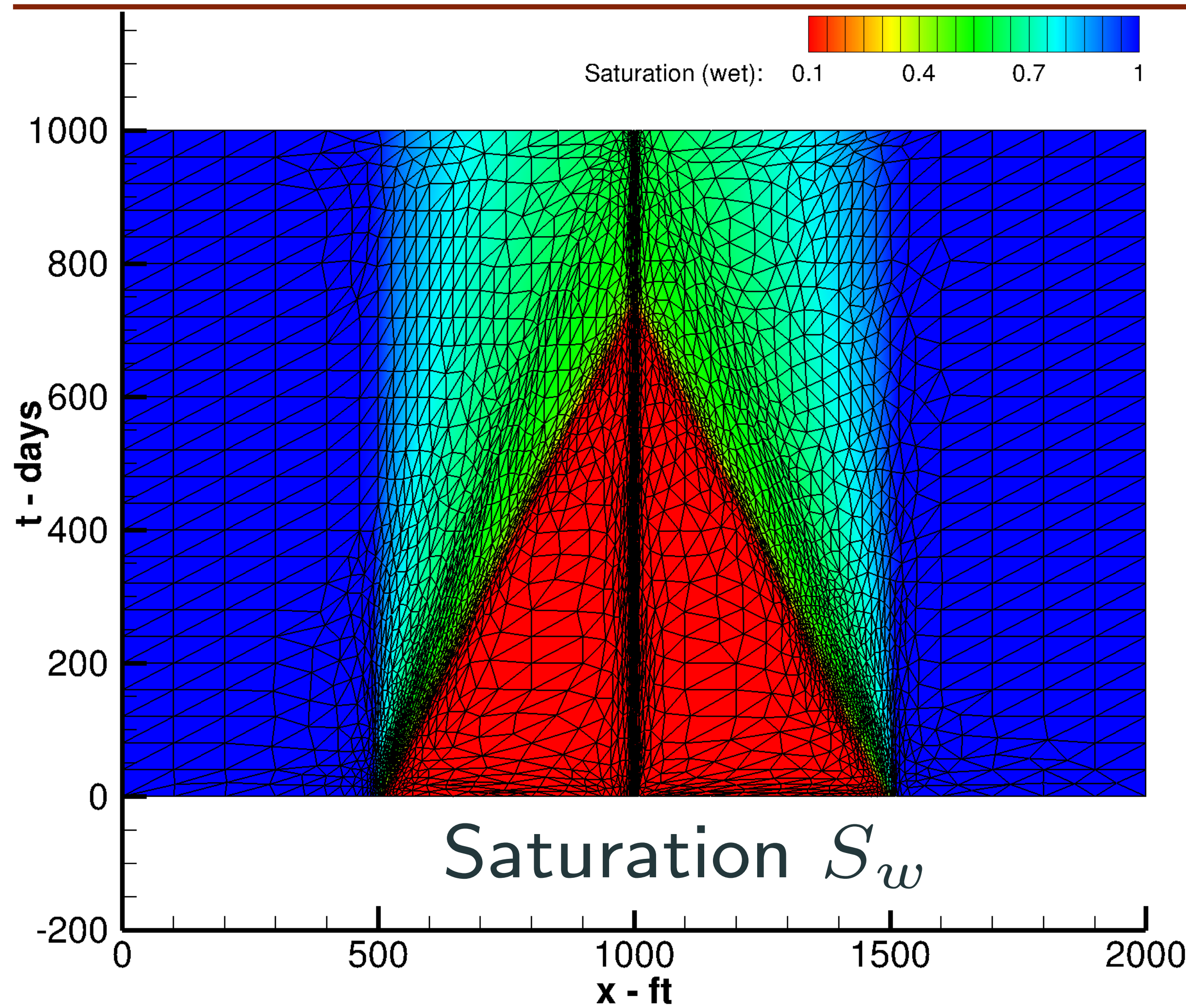

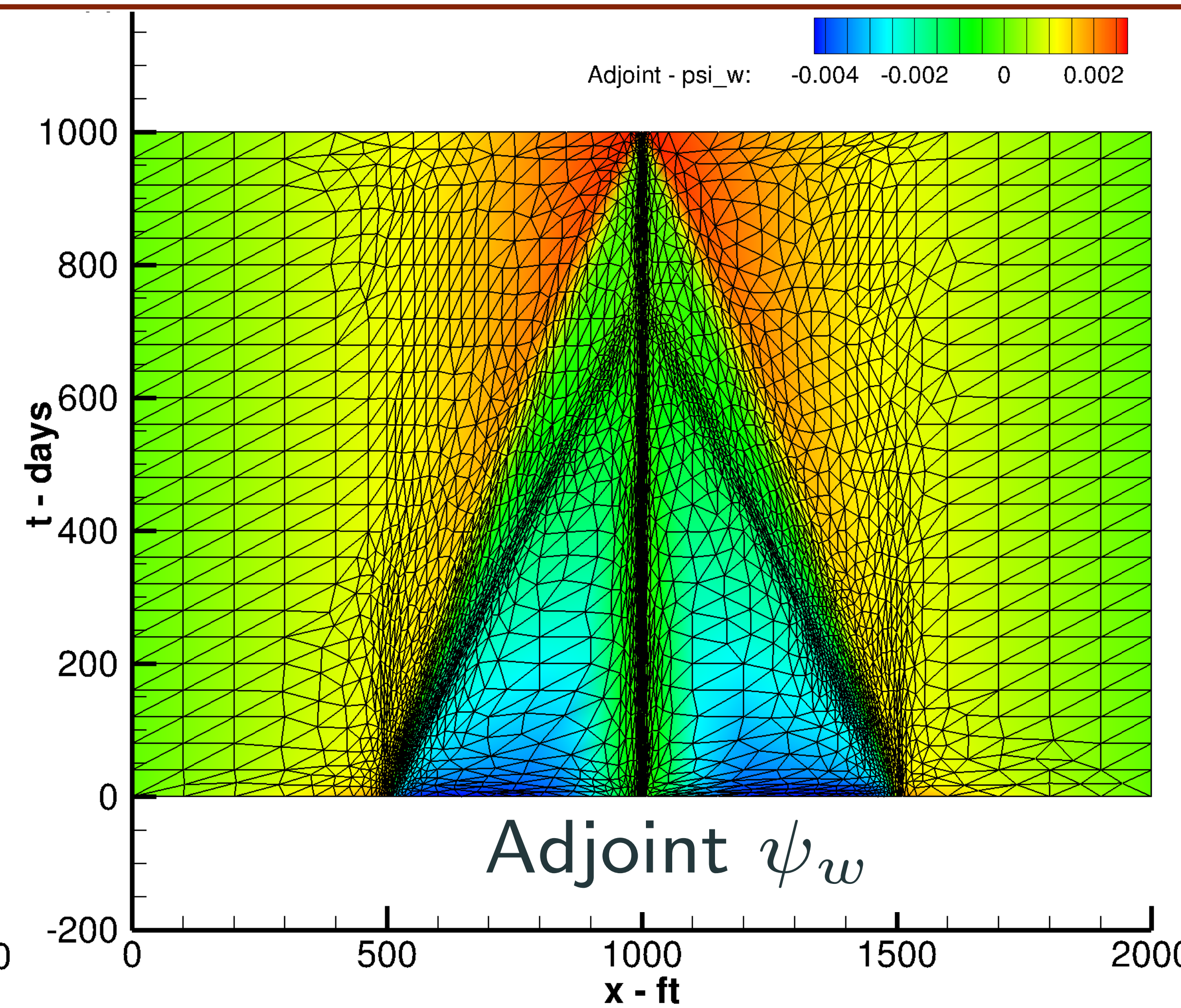

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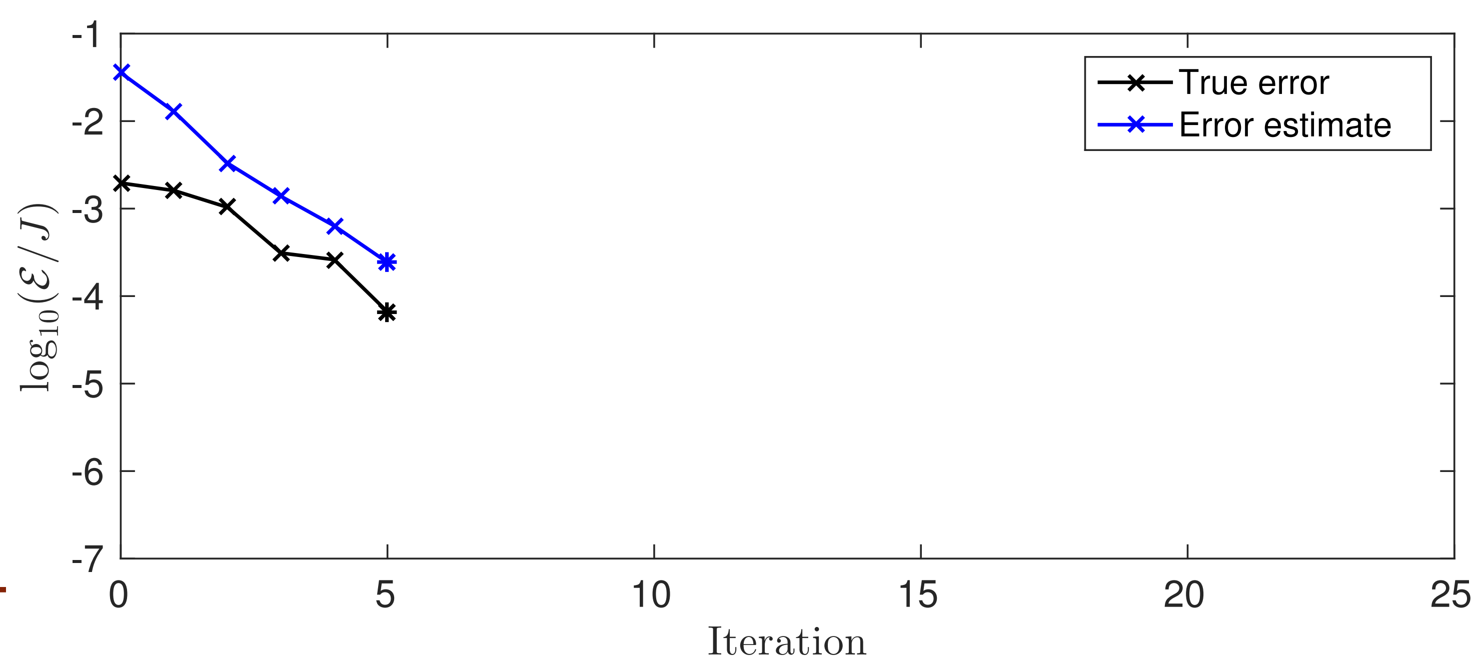

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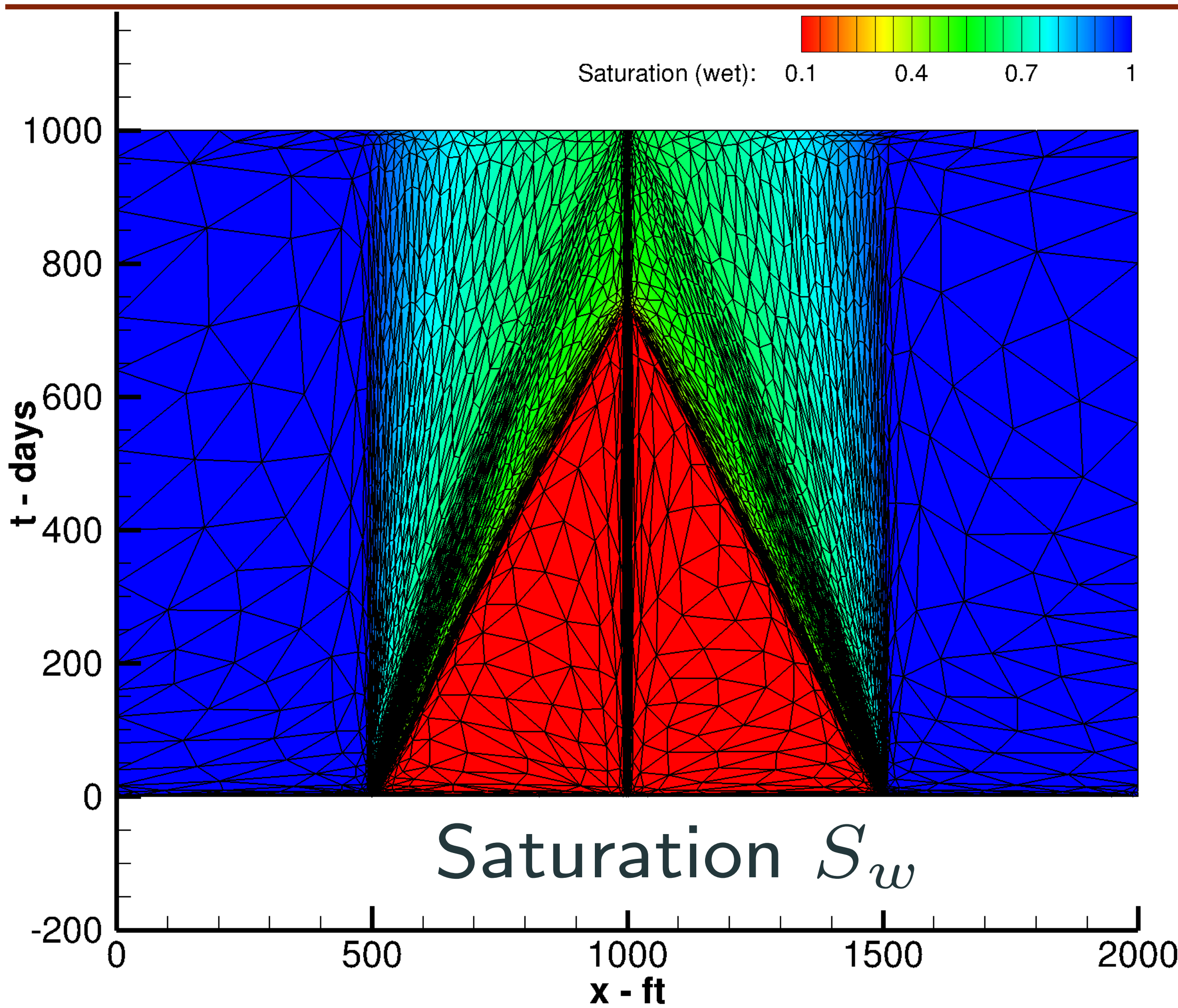

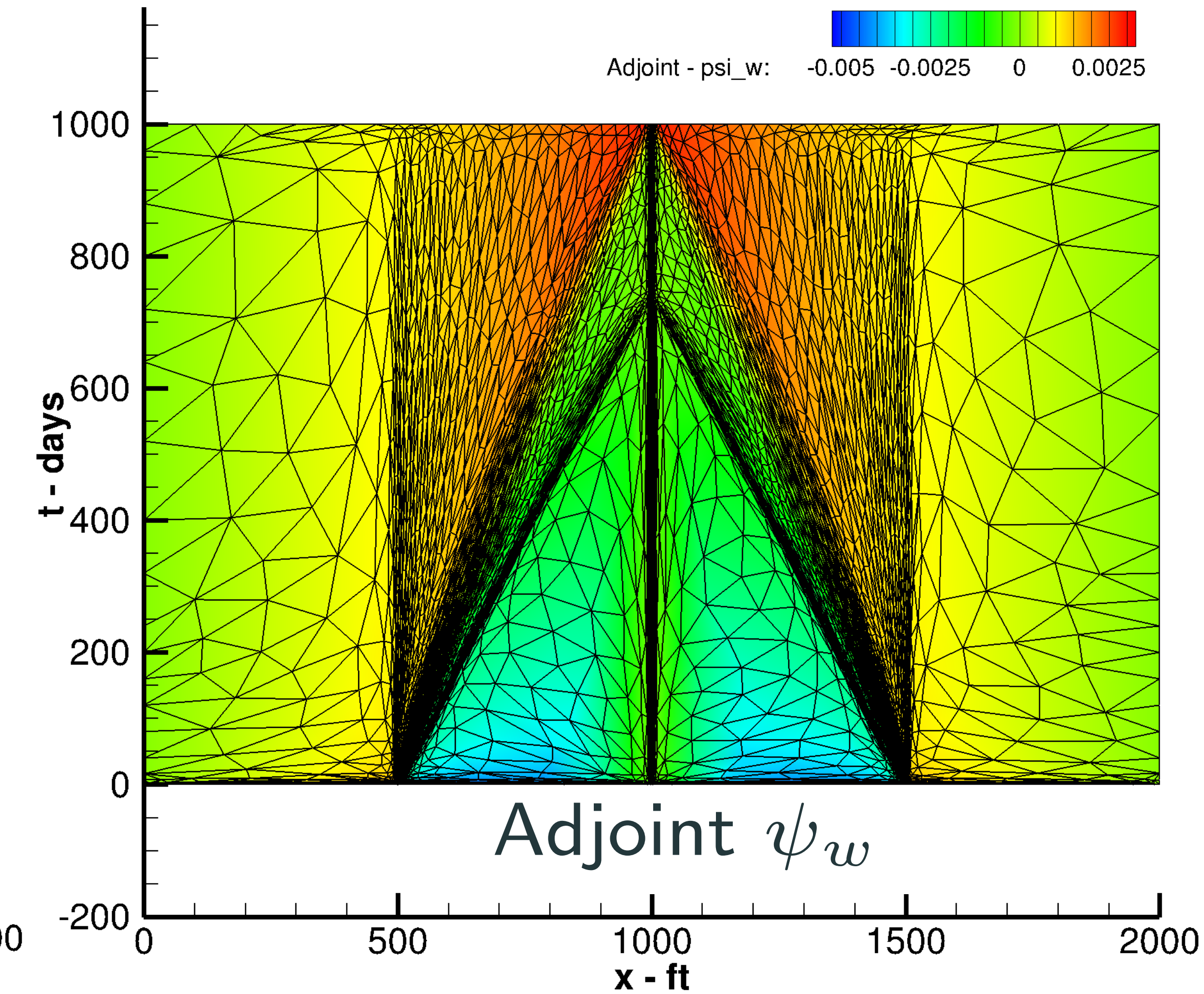

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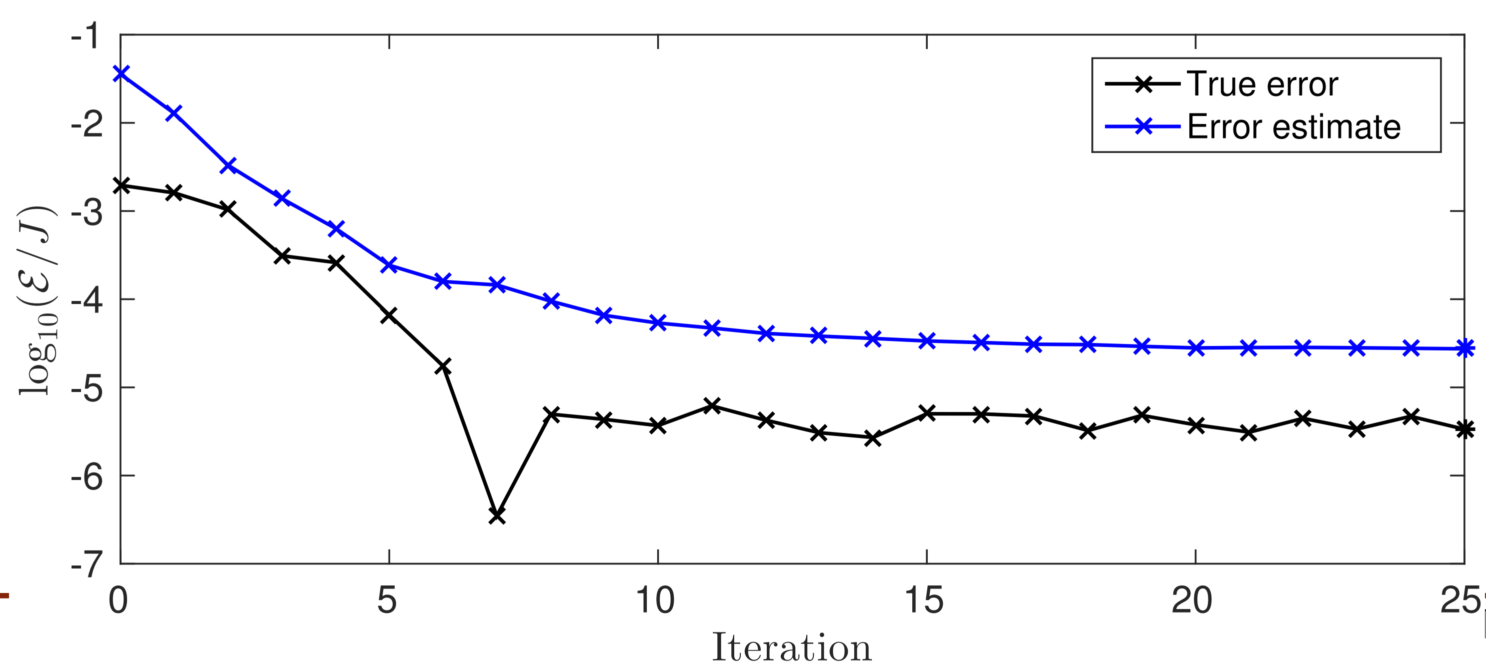

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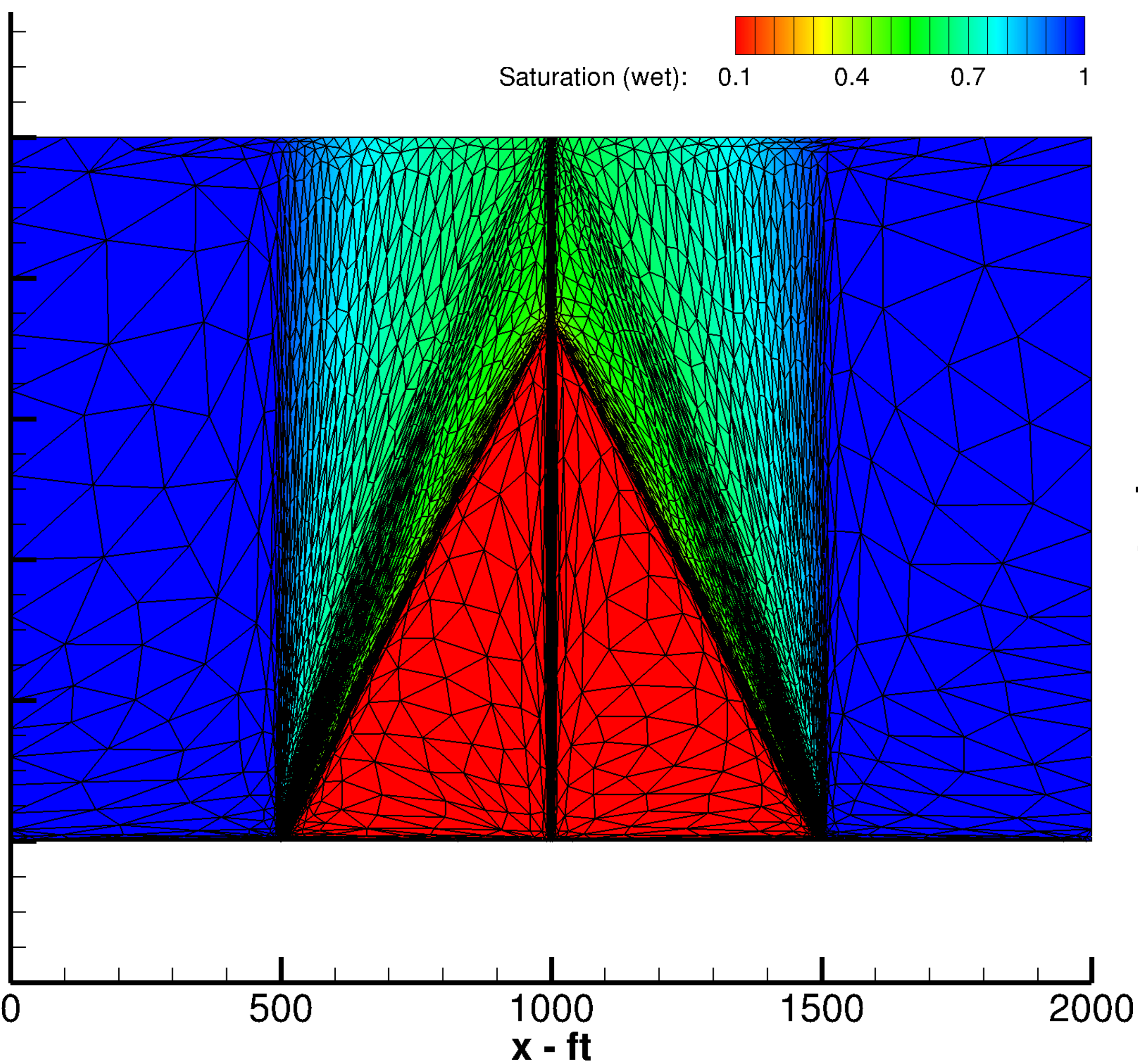

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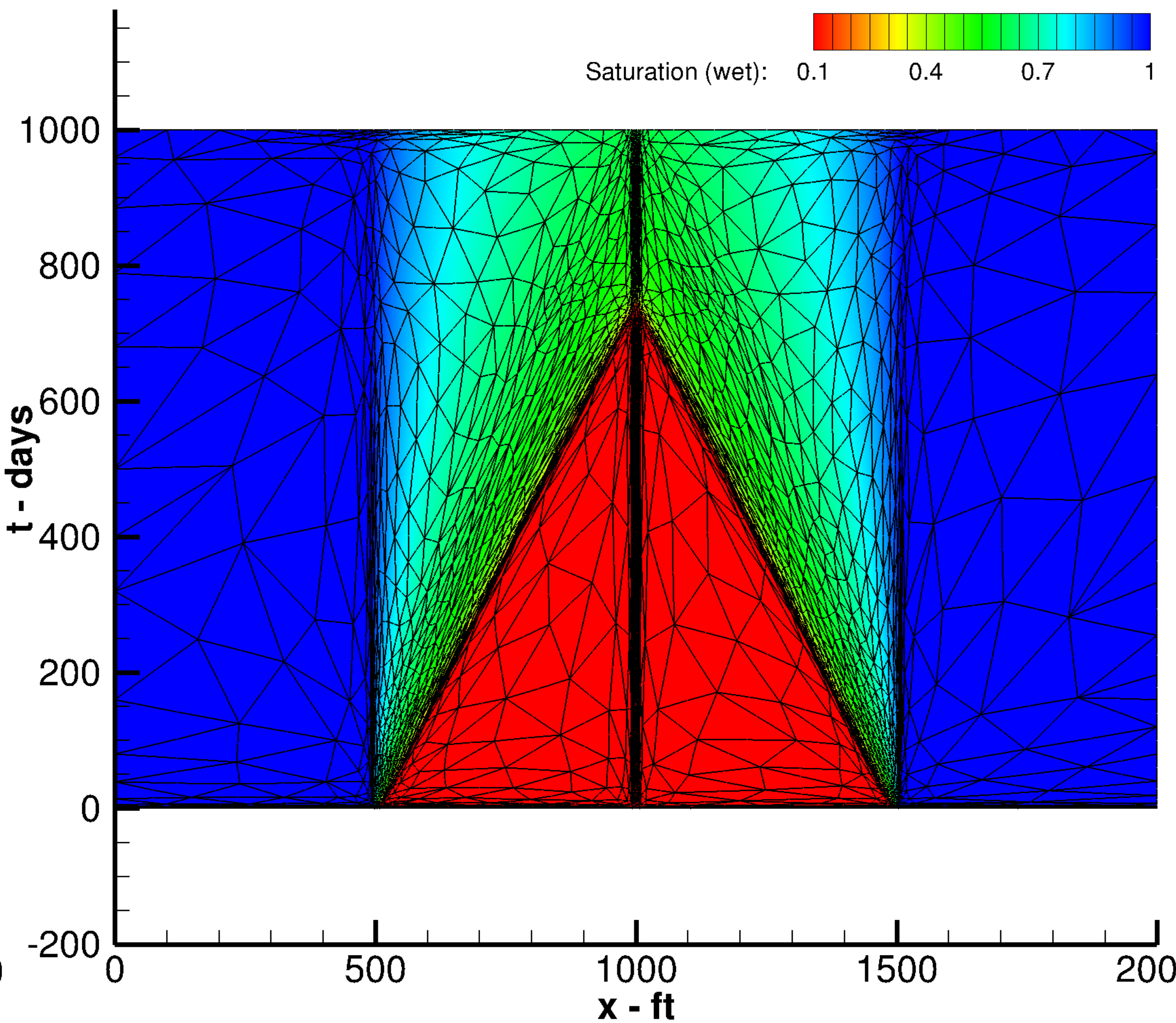

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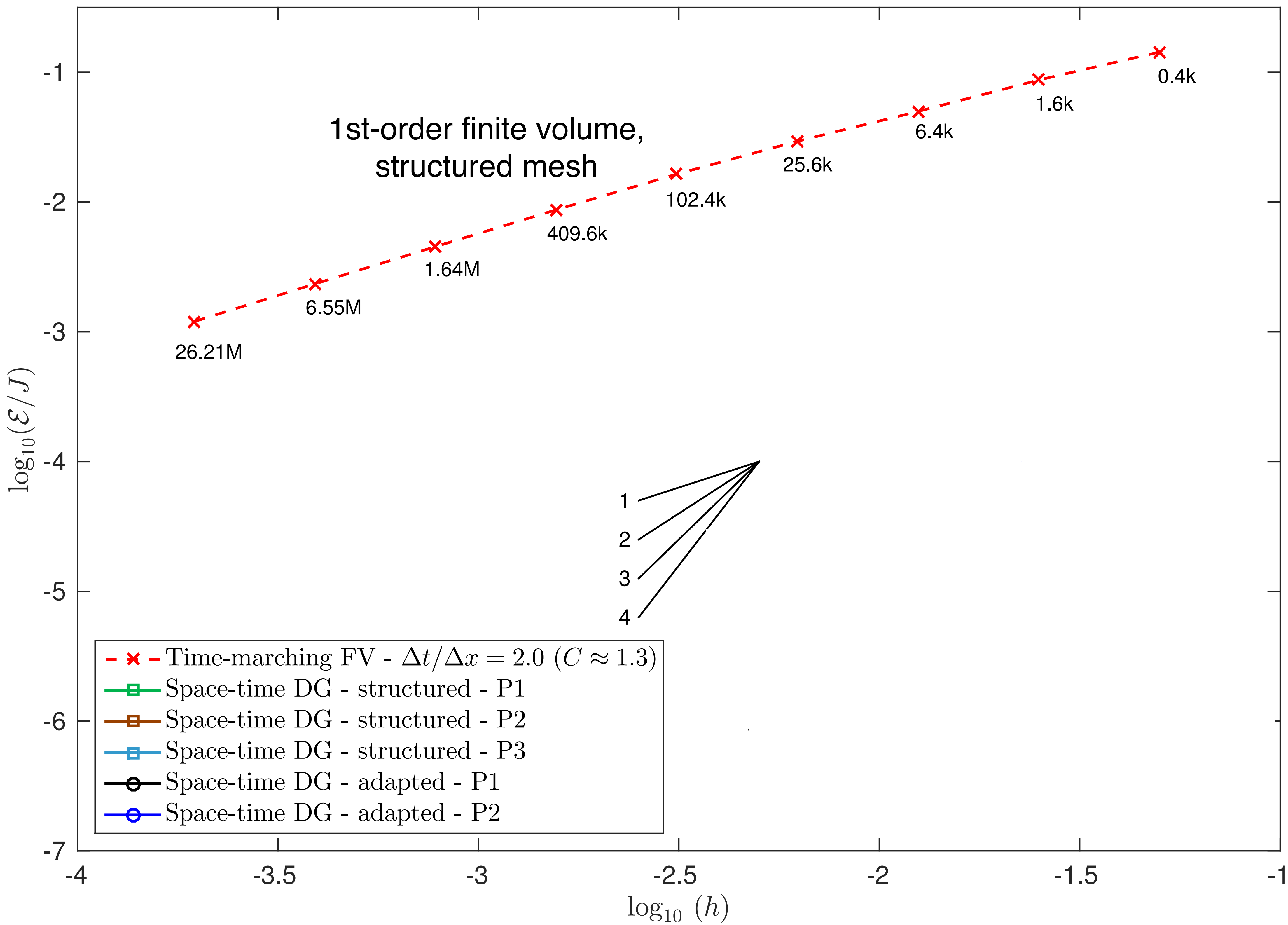
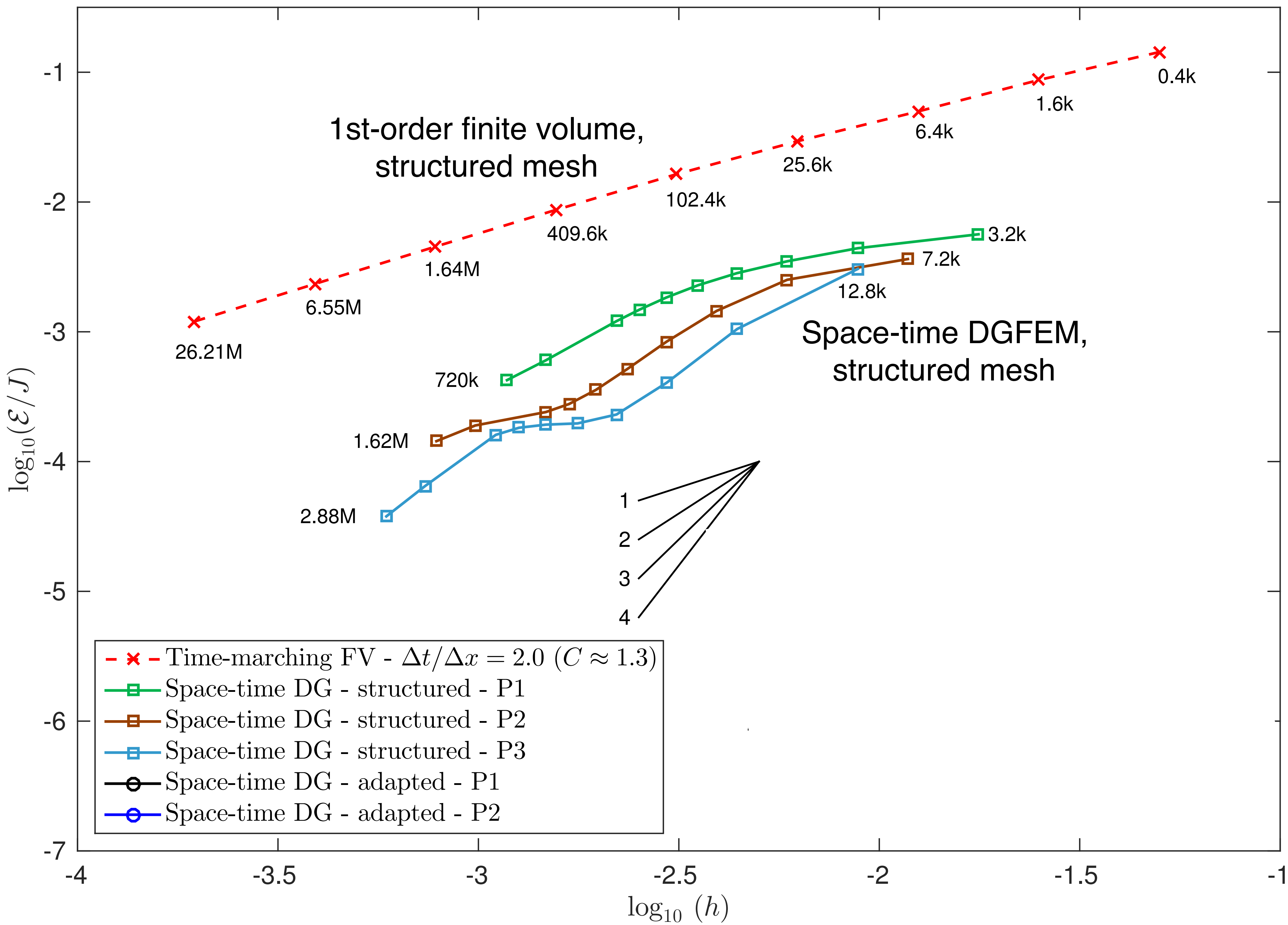
Application: Two-phase porous media flow



Application: Two-phase porous media flow



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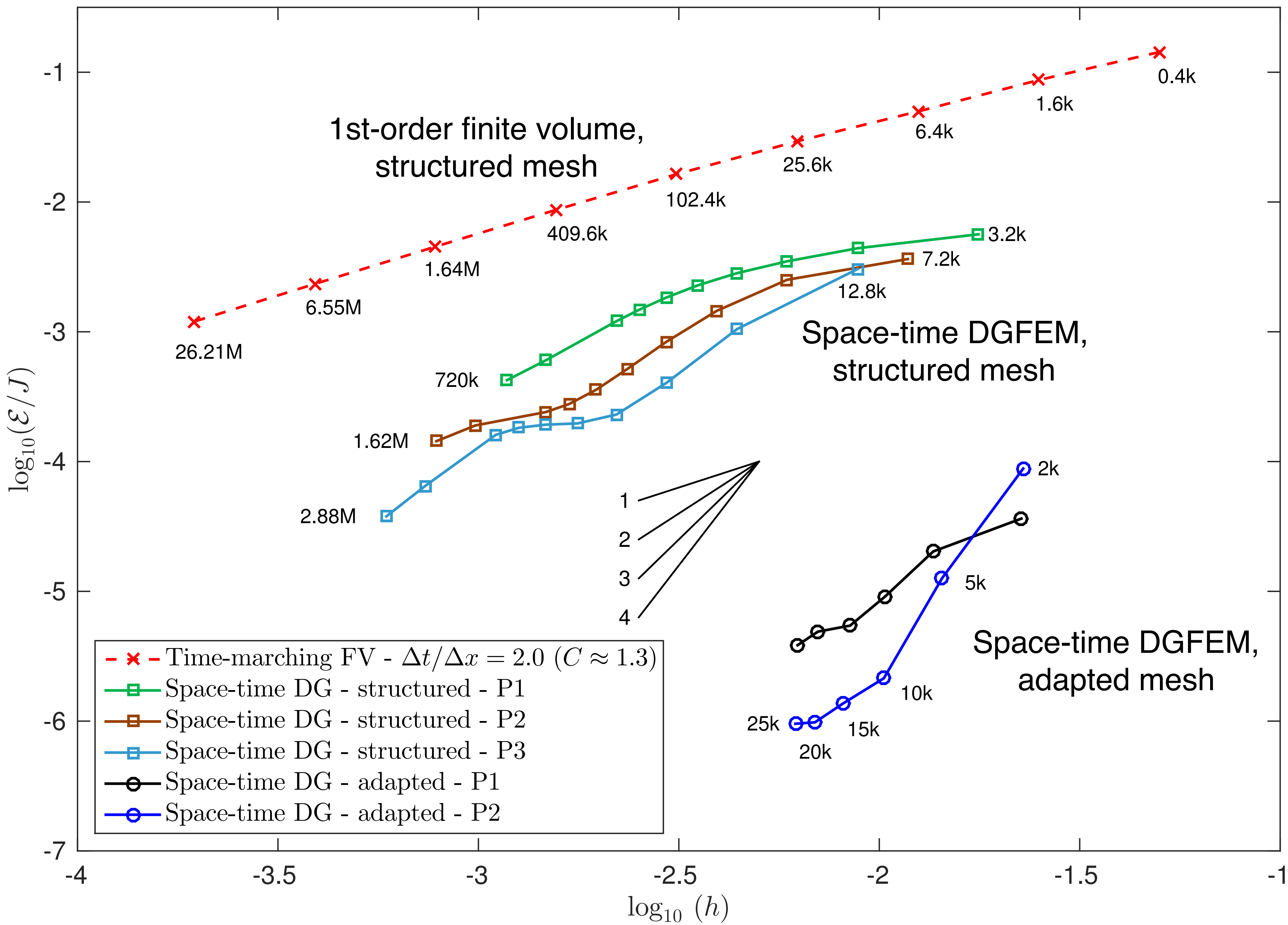

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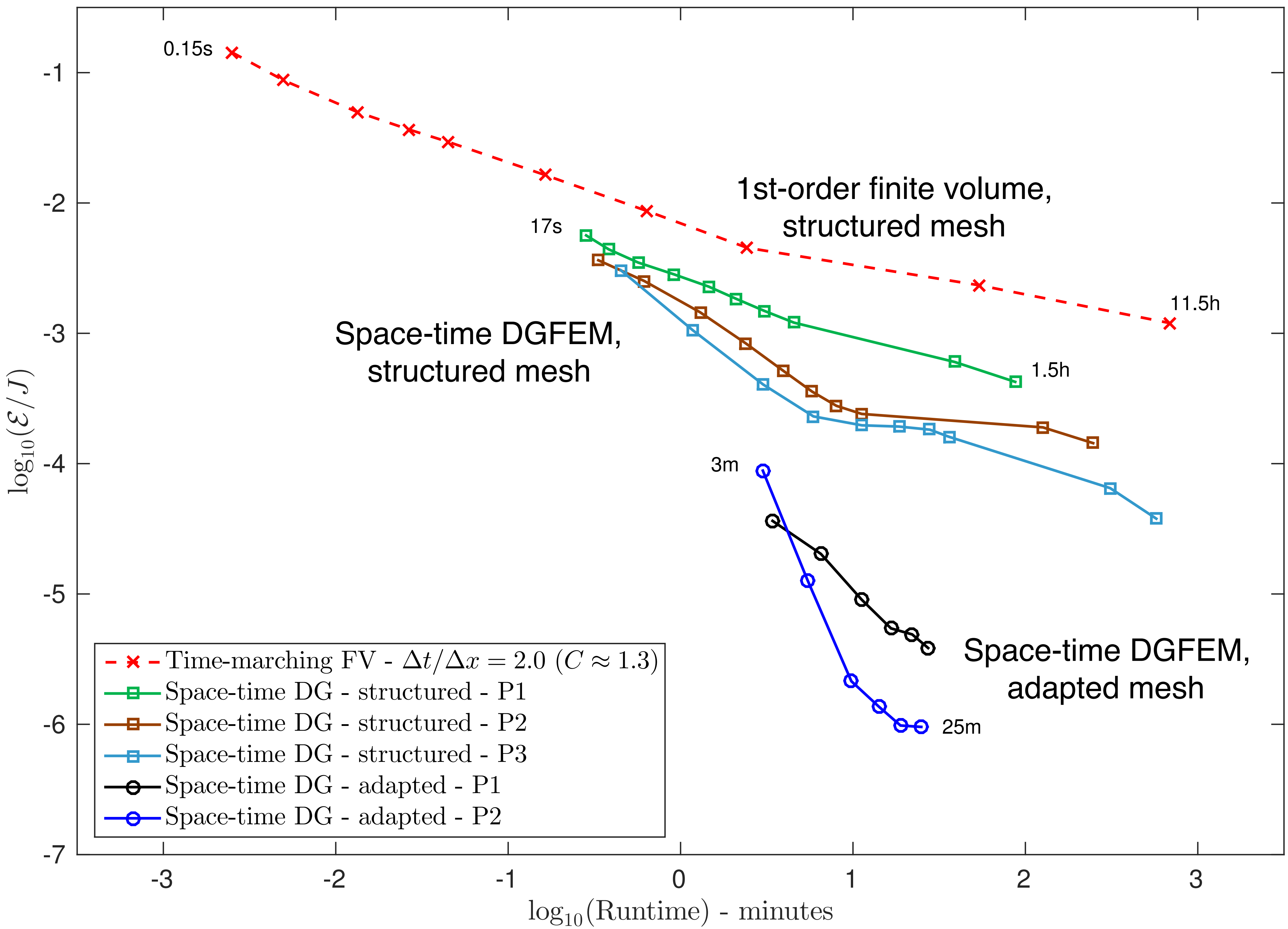

Application:
2D shedding cylinder test case

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- $Re = 100, M = 0.1$
- Smooth initial conditions (Higher-order Workshop test case)

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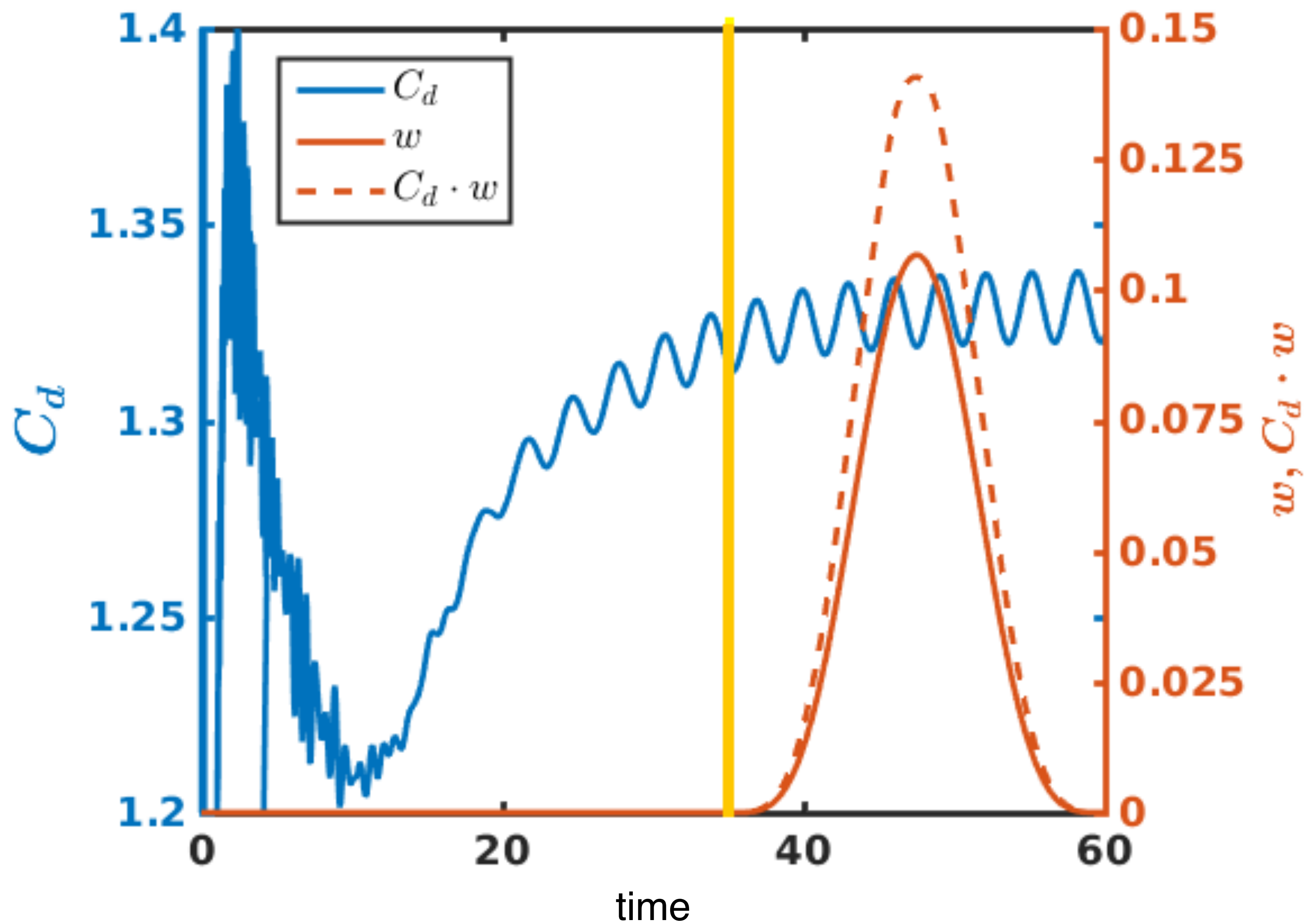
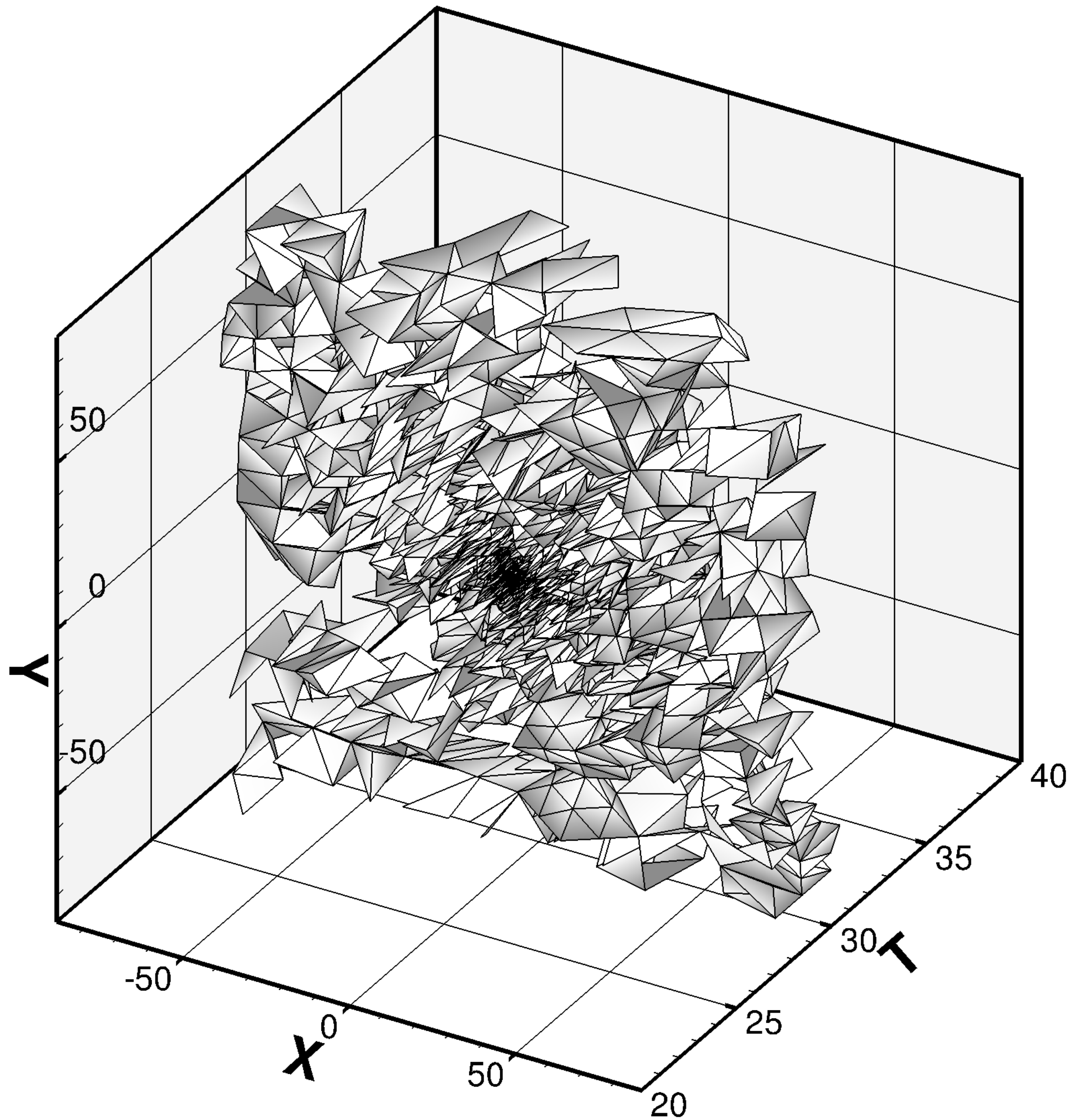
- $Re = 100, M = 0.1$
- Smooth initial conditions (Higher-order Workshop test case)
- Output:

$$J = \int_0^T C_d(t) w(t)\, dt$$

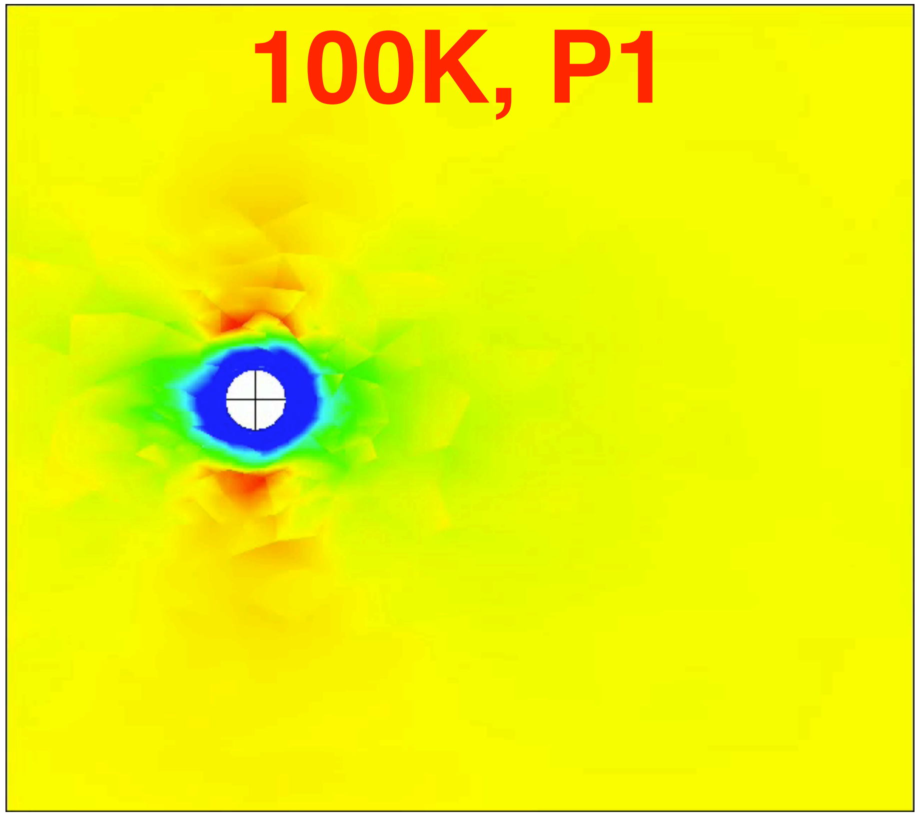
$w(t) =$ Hann-squared window (Krakos et al 2012)

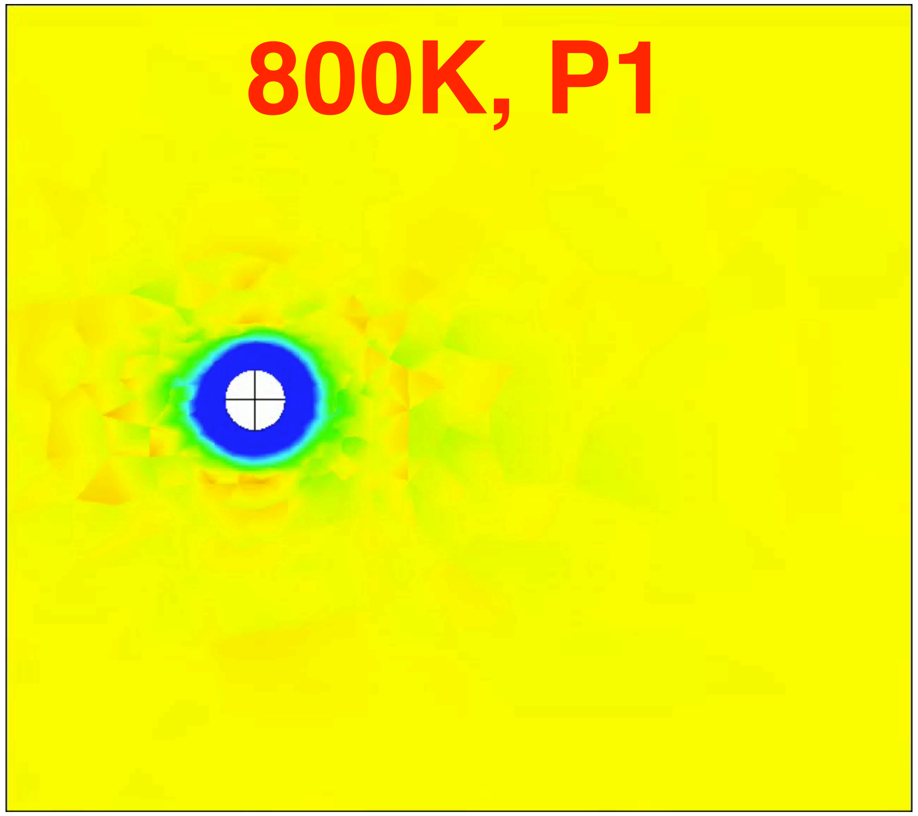


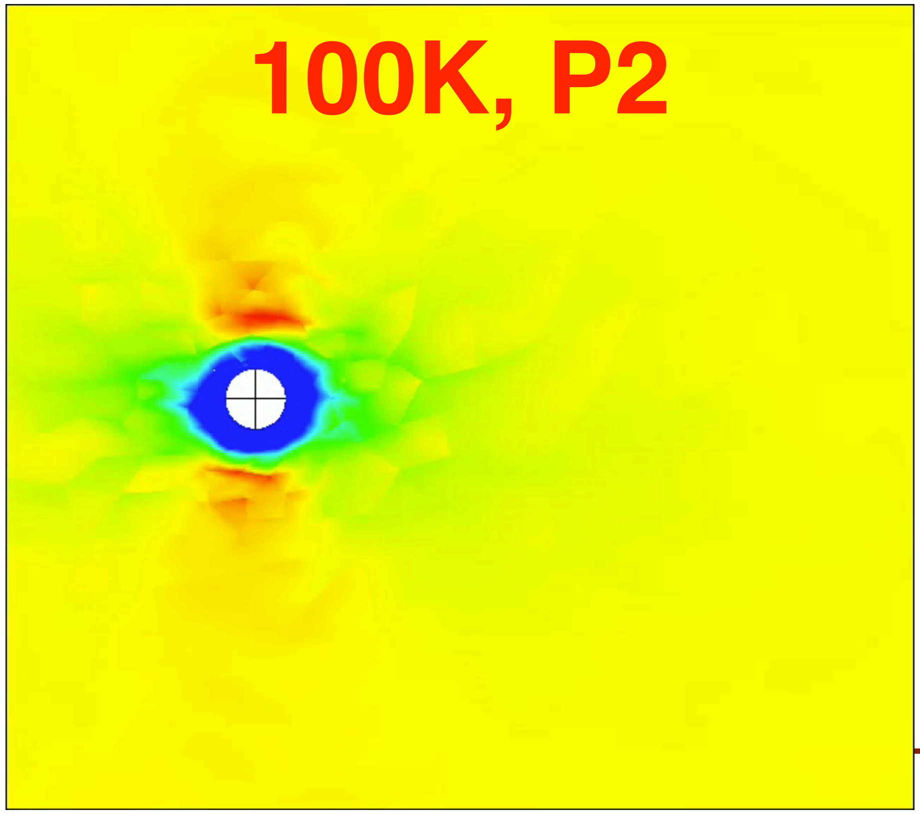
2D circular cylinder test case: Typical adapted space-time mesh

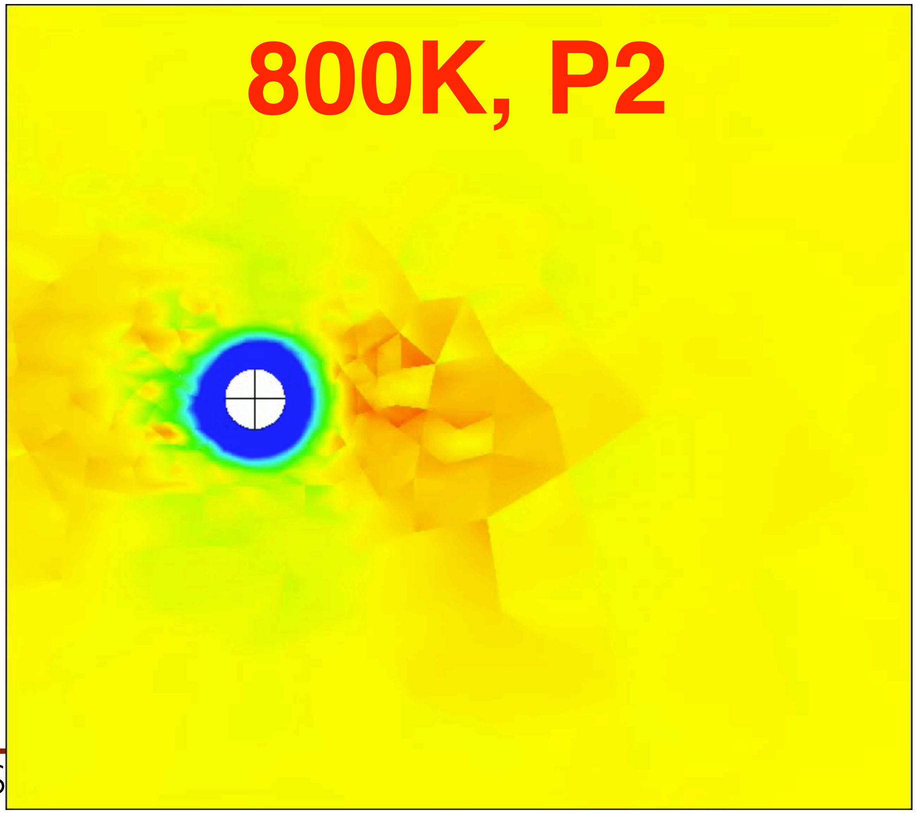


Space-time adaptive solutions

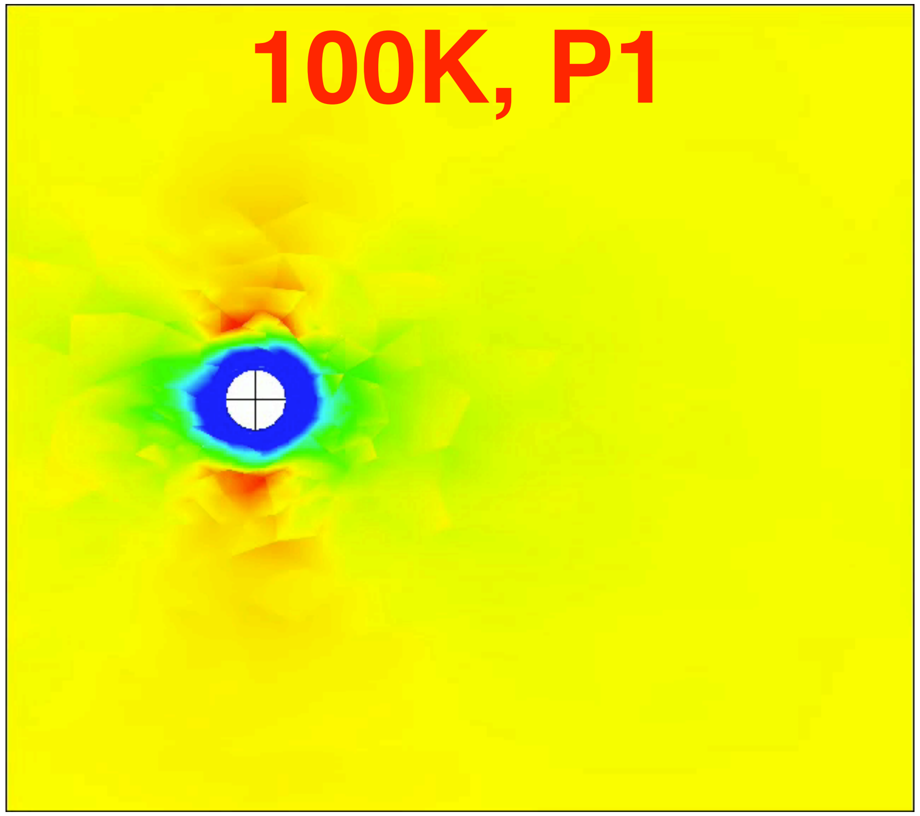


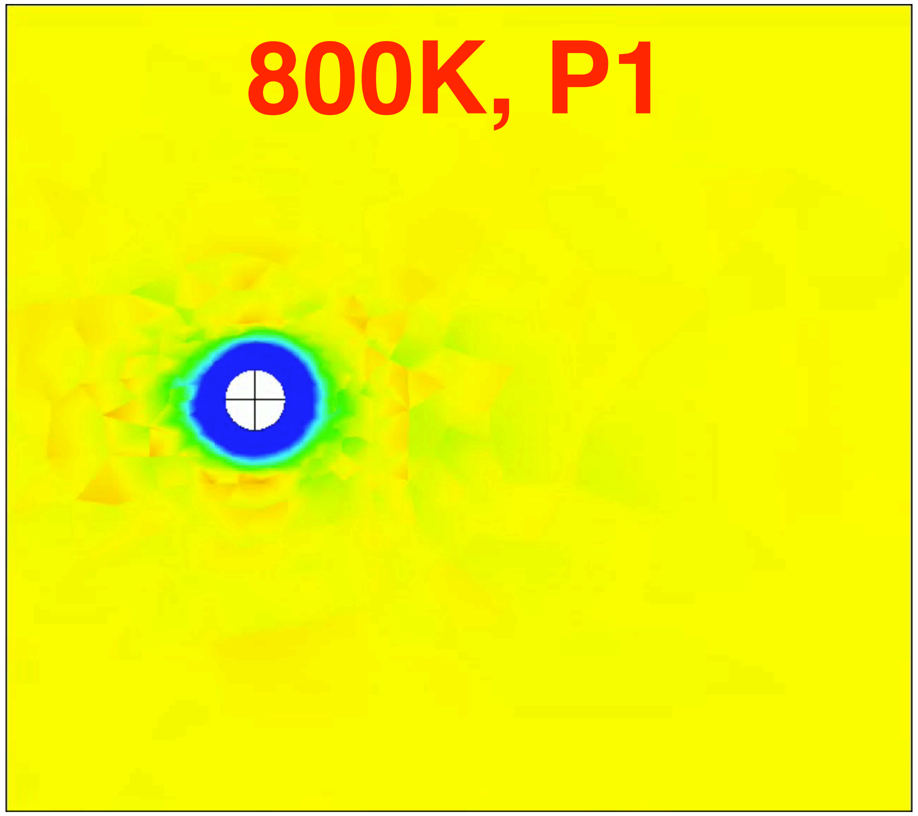


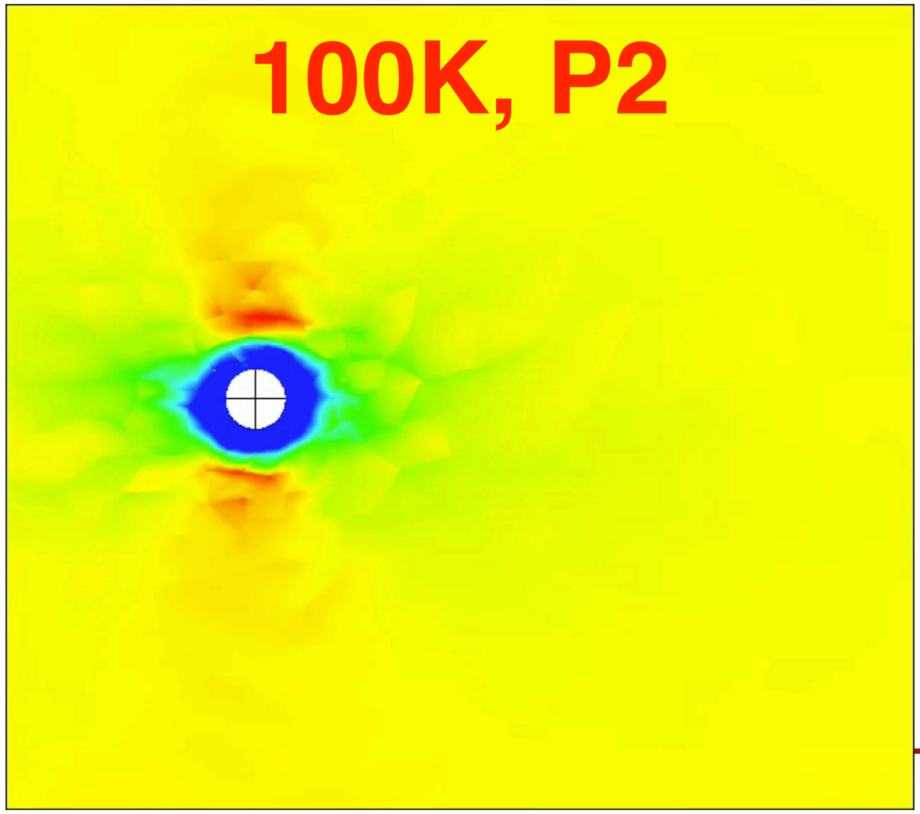


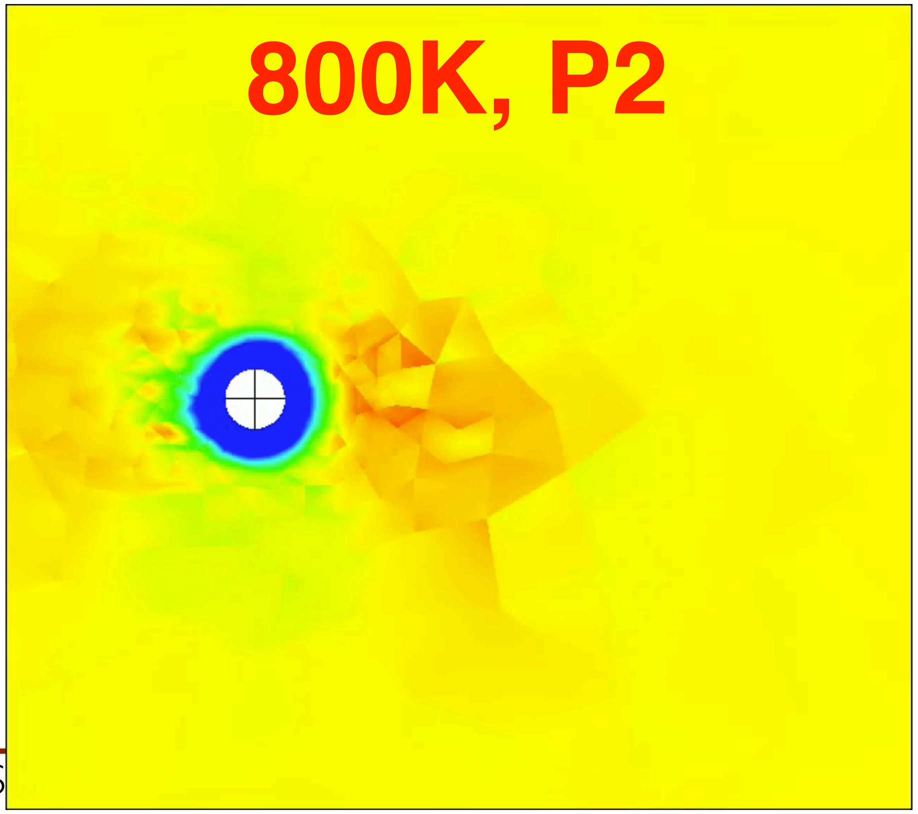


Space-time adaptive solutions



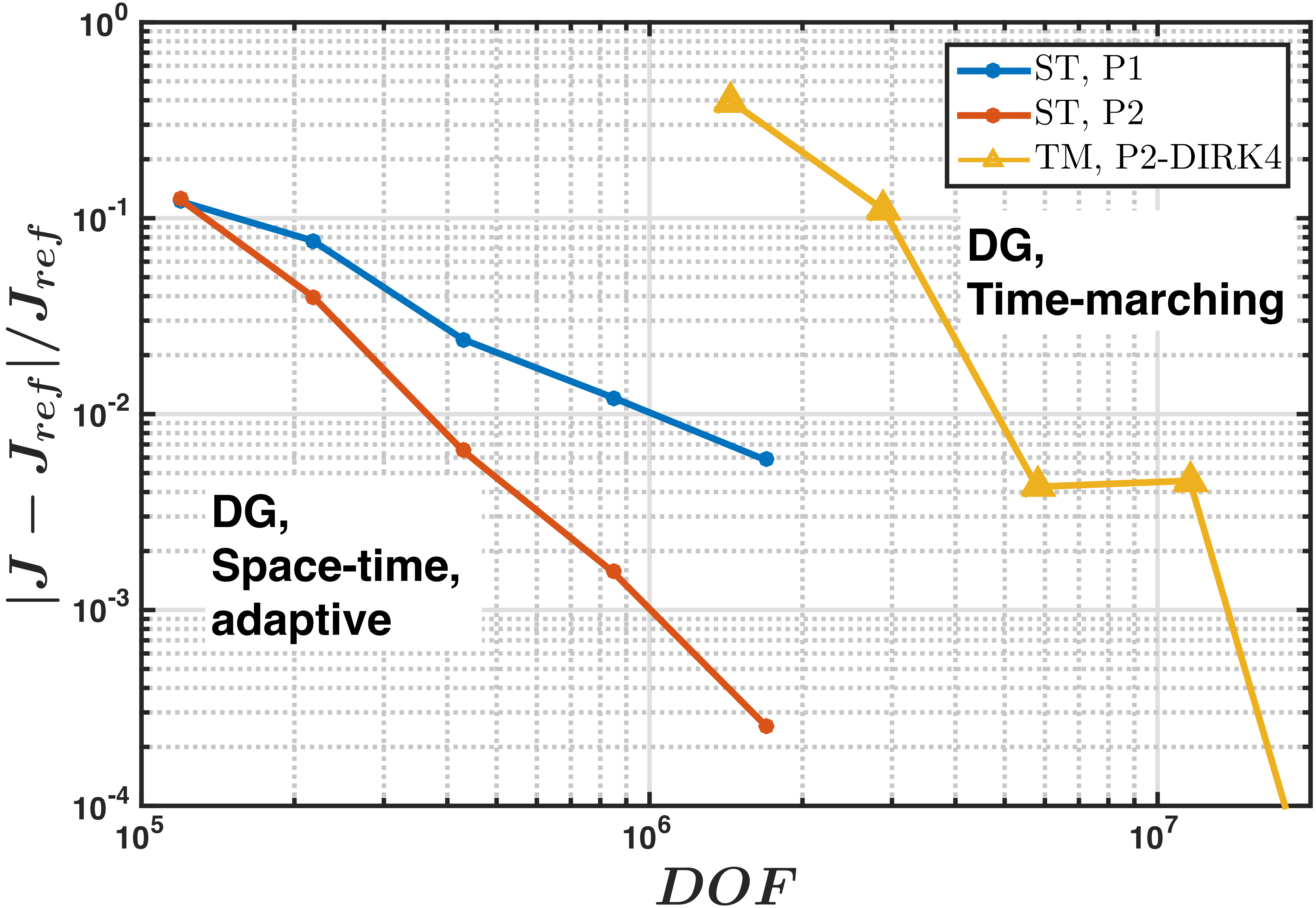






Drag versus DOF:

Time-marching vs space-time adaptive



On-going Work

- Solution methods for space-time adaptive meshes (leverage hyperbolic time dependence)
- 4D adaptive meshing for 3D space + time

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Questions?