Incremental progress towards hexahedral mesh generation

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2D mesh singularity points

- **Structured quad mesh**
  - Four elements meeting at a internal node
  - Grid topology + boundary alignment → too restrictive

- **Block-structured quad mesh**
  - Simple blocks meeting at nodes of irregular connectivity, i.e. mesh singularities
  - Positive singularity: > 4 elements, negative singularity: <4 elements

Positive singularity

Negative singularity
3D line singularities

- Singularities travels from one face to another
  - Sweeps, multi-sweeps, thin sheets, long slender regions
- Singularities forms loops
  - Revolves
- Singularities meet
  - Limited number of patterns

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Volume decomposition

• Strategy
  – Thin sheet, long slender and residual regions
    • Reduce the decomposition effort
    • Reduce the DOF of the analysis model

Residual complex regions - mesh patterns at vertices

- Optimum element number $n_i$ at corner angle $\theta_i$

$$n_i = \text{round} \left( \frac{\theta_i}{\pi/2} \right)$$

$n_c$:

<table>
<thead>
<tr>
<th>$n_c$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimum $\theta_i$ range:</td>
<td>$[0, \frac{\pi}{4})$</td>
<td>$[\frac{\pi}{4}, \frac{3\pi}{4})$</td>
<td>$[\frac{3\pi}{4}, \frac{5\pi}{4})$</td>
<td>$[\frac{5\pi}{4}, \frac{7\pi}{4})$</td>
<td>$[\frac{7\pi}{4}, 2\pi]$</td>
</tr>
</tbody>
</table>
Identifying surface singularities

$$\sum_{\text{vertices}} \left( \frac{\pi}{2} (n_i - 2) + \alpha_i \right) + \sum_{\text{edges}} \int k_g \, ds + \iint_{\text{face}} K \, dS + (n_+ - n_-) \frac{\pi}{2} = 0$$

A continuum theory for unstructured mesh generation in two dimensions, G Bunin, CAGD, vol. 25, 14–40, 2008

Euler Characteristic

$$\chi = V - E + F$$

where $V$, $E$, $F$ are the number of vertices, edges and faces of any subdivision of the surface

Gauss-Bonnet theorem

$$\oint_{\partial R} k_g \, ds + \iint_R K \, dS + \sum_{i=1}^{N} \alpha_i = 2\pi \chi$$

$N$: the number of corners

$n_i$: no of elements at each corner
Singularities on simple surfaces

\[ n_+ - n_- = -4\chi + \sum_{i=1}^{N} (2 - n_i) \]
Controlling Mesh Density

- Extra singularity pair (a dislocation)
Finding required number of surface singularities

\[ n_+ - n_- = -4\chi + \sum_{i=1}^{N} (2 - n_i) \]

\( \chi = 2, N = 0, n_+ - n_- = -8 \)

\( \chi = 0, N = 0, n_+ - n_- = 0 \)
Applications

• Identify sweep-able volumes \[1\]
  – No mesh singularities on wall faces

• Revolves

Surface singularities in the 3D residual regions

\[ n_+ - n_- = -1 \]
\[ n_+ - n_- = 0 \]
\[ n_+ - n_- = +1 \]
\[ n_+ - n_- > 1 \text{ or } n_+ - n_- < -1 \]
Singularity placement: using offsets

- Locate the position of the singularities
  - When the number of singularities changes after offset, a singularity should be placed
Singularity placement: offsets vs medial axis$^{[1]}$

- Locate the position of the singularities
  - When the included angle between medial radii changes

Singularity placement: medial axis vs offset

Singularities calculated by making offset of the boundary

Singularities calculated from medial axis
Singularity placement: medial axis vs cross field

- Advancing front of crosses
- Singularities occur where MAT changes from aligned with mesh to diagonal
- Can handle
  - Variations in target element size, shape and orientation
  - Large differences in feature size
- Doesn’t need precise MAT, but singularities end up in very similar places for isotropic elements

Conclusions

• Placement of mesh singularities is key to structured multi-block hex meshing
• Thin sheets: singularities start on one surface and exit on the opposite one
• Long slender regions: singularities run from source to target faces
• Multi-sweep regions: similar
• Revolves: singularities form a loop
• Residual complex regions
  – Simple analysis using Euler characteristic and number of elements at each corner provides minimum necessary number of singularities emerging on each face
  – Can add addition positive/negative singularity pairs to provide target mesh size distribution
• An incremental approach helps identify strategies for different singularity patterns
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